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Almost periodic movements in uniform spaces

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ALMOST PERIODIC MOVEMENTS IN UNIFORM SPACES

by

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I. INTRODUCTION

The theory of almost periodic functions was initiated by Harald Bohr in three long papers [1][1] published in Acta Mathematica during the years 1924-1926. From 1924 until his death in 1951 Harald Bohr developed further details of this theory in a total of fifty-four papers [10] written in Danish, German, French and English.

Bohr's theory of almost periodic functions is restricted to the class of complex-valued functions f(t) defined and continuous for all real values of t. During the years following Bohr's 1924 paper various generalizations of Bohr's theory to the larger class of Lebesgue measurable functions were developed by A. S. Besicovitch [2], W. Stepanoff [3], H. Weyl [4], and others including Bohr himself [10]. Consequently Bohr's original amost periodic functions have come to be called "ordinary almost periodic functions" in order to distinguish them from these generalizations. However, since the only almost period functions dealt with in this thesis (with one minor exception) are continuous functions of a real variable, we shall refer to Bohr's complex-valued continuous almost periodic functions of a real variable simply as "almost periodic functions", and

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[1]Numbers in square brackets refer to the list of references at the end of the thesis.

use special adjectives for almost periodic functions having their ranges in something other than the space of complex numbers.

One of the main results of the theory of almost periodic functions is the "approximation theorem" which states that any almost periodic function can be uniformly approximated by a trigonometric polynomial. More precisely, if $f(t)$ is an almost periodic function, then given any positive number ϵ there exist complex numbers A_k, $k = 1,\ldots, N$ and a set of real numbers β_k, $k = 1,\ldots, N$ such that

$$\left| f(t) - \sum_{k=1}^{N} A_k e^{i\beta_k t} \right| \leqq \epsilon$$

for all real t. Analogous approximation theorems have been established for all the various generalizations of almost periodic functions. Quite recently H. Tornehave [11] introduced the concept of an "almost periodic movement" which is essentially a Bohr-type almost periodic function of a real variable, the main difference being that its values lie in an "arbitrary" metric space M, rather than in the space of complex numbers. Tornehave established an approximation theorem for certain "families" of almost periodic movements $f(t;v)$ (where v is the parameter of the family) which may be stated roughly as follows: To each "family" $f(t;v)$ and each positive number ϵ there correspond positive integers m and N, a continuous function $g(u_1,\ldots,u_m;v)$,

$-\infty < u_\nu < +\infty$, $\nu = 1,\ldots,m$ with the period $2N\pi$ in each of the variables u_ν, and real numbers $\beta_1,\ldots,\beta_m$ such that

$$d(g\,(\beta_1 t,\ldots,\beta_m t;v),\ f(t;v)) \leqq \epsilon$$

for all real t and all v. Here d(x,y) denotes the distance between the points x and y of the metric space M.

In Chapter III of this thesis the concepts of Tornehave are generalized to the case where M is a uniform space, and the corresponding approximation theorem is established. Perhaps the most notable examples of uniform spaces are metric spaces, topological groups, and topological vector spaces. The notion of a uniform space, introduced by A. Weil [8] in 1937, is essentially a special type of general topological space in which there is "just enough" additional "structure" to render definable such concepts as uniform continuity, uniform convergence, uniform approximation, Cauchy sequence, total boundedness, and completeness. Since these are precisely the "topological" concepts upon which Bohr's theory of almost periodic functions is based, it seems that perhaps a uniform space is the most general range-space possible for a Bohr-type theory of almost periodic functions. Indeed, S. Bochner and J. von Neumann [7] have already generalized Bohr's theory to the case where the functions concerned are defined on an arbitrary group with values in a complete convex topological vector space.

Only that part of the generalization of Tornehave's theory which is necessary to establish the approximation theorem has been given in this thesis. After the concepts are generalized in an appropriate fashion the proofs of most of the theorems can be carried through in the same way as in Tornehave's work with only minor changes to adapt them to the more general structure of a uniform space. Those theorems which are different in some respect, or whose proofs required more than just minor changes, are marked by an asterisk. It was found that the approximation theorem could be established with a somewhat weaker concept of completeness than is usually given for a uniform space. Accordingly some theorems concerning the adopted notion of completeness in a uniform space are presented in Chapter III, Section B, just before they are first needed.

Chapter II of the thesis contains the necessary preliminaries regarding Bohr's theory of almost period functions, topological spaces, and uniform spaces. Proofs of theorems and further information concerning these theories can be found in A. S. Besicovitch's book [6] and John L. Kelley's book [12]. For further information regarding almost periodic sequences the reader is referred to the paper [5] by Ingeborg Seynsche, and to the references cited there.

Finally, it was found that one of Tornehave's theorems, concerning a uniformly continuous family of almost periodic movements, leads to an interesting necessary and sufficient condition that a given solution $\phi(t)$ of a differential equation of the form

$$\frac{dx}{dt} = F(x,t)$$

be an almost periodic function. A precise statement of this result, and its proof, is the content of Chapter IV of this thesis, which can be read independently of Chapter III. The necessary preliminaries for Chapter IV are contained in Chapter II, Section A.

II. PRELIMINARIES

A. Almost Periodic Functions

The following definitions and theorems from the classical Bohr-theory of almost periodic functions are needed for Chapters III and IV of this thesis. The material following Theorem 4 is not used in Chapter IV.

Definition 1. A set E of real numbers is called relatively dense (r.d.) if there exists a positive number l such that every interval of length l contains at least one element of the set E.

Definition 2. Let f(t) be a complex-valued function defined for all real t. If a real number τ satisfies the inequality

$$|f(t+\tau) - f(t)| \leqq \epsilon \qquad \text{for all real } t,$$

then τ is called an ϵ-translation number of f(t). If τ is an integer then it is called an ϵ-translation integer of f(t). The set of all ϵ-translation numbers of f(t) is denoted by $\{\tau_f(\epsilon)\}$.

Definition 3. Let a(n) be a complex-valued function defined on the integers $n = 0, \pm 1, \pm 2, \ldots$. If an integer τ satisfies the inequality

$$|a(n+\tau) - a(n)| \leqq \epsilon \quad \text{for all } n = 0, \pm 1, \pm 2, \ldots,$$

then τ is called an <u>ϵ-translation</u> <u>integer</u> of a(n). The set of all ϵ-translation integers of a(n) is denoted by $\{\tau_a(\epsilon)\}$.

Definition 4. A complex-valued function f(t) defined and continuous for all real t is called an <u>almost</u> <u>periodic</u> (<u>a</u>.<u>p</u>.) <u>function</u> if for each positive number ϵ the set of ϵ-translation numbers of f(t) is relatively dense.

Definition 5. A complex-valued function a(n) defined on the integers n = 0, ± 1, ± 2,... is called an <u>almost</u> <u>periodic</u> <u>sequence</u> if for each positive number ϵ the set of ϵ-translation integers of a(n) is relatively dense.

Theorem 1. If f(t) is an almost periodic function, then for each positive number ϵ the set of ϵ-translation integers of f(t) is a relatively dense set.

Corollary. Every almost periodic function, restricted to the integers, is an almost periodic sequence.

Theorem 2. Every almost periodic function (a.p. sequence) is bounded.

Theorem 3. Every almost periodic function is uniformly continuous on the entire real line.

Theorem 4. The sum of two almost periodic functions (a.p. sequences) is again an almost periodic function (a.p. sequence, respectively).

Definition 6. A finite or denumerable set of real numbers $\{\alpha_i : i = 1,2,\ldots\}$ is called <u>linearly</u> <u>independent</u> (= <u>rationally</u> <u>independent</u>) if for each n the only rational values of $r_1, r_2, \ldots r_n$ satisfying the equation

$$r_1\alpha_1 + r_2\alpha_2 + \ldots + r_n\alpha_n = 0 ,$$

are $r_1 = r_2 = \ldots = r_n = 0$.

Theorem 5. Given an arbitrary finite or denumerable set of real numbers $\{a_i : i = 1, 2, \ldots\}$ there exists a finite or denumerable set of linearly independent numbers $\{\alpha_i : i = 1, 2, \ldots\}$ such that each a_n can be represented as a finite linear combination of α's with rational coefficients in the form

$$a_n = r_{n1}\alpha_1 + \ldots + r_{nq_n}\alpha_{q_n} .$$

Theorem 6. (Kronecker's Theorem). If $\alpha_1, \alpha_2, \ldots, \alpha_k$ are k linearly independent real numbers, $a_1, a_2, \ldots, a_k$ are k arbitrary real numbers, and T and ϵ are positive, then there is a real number t, and integers $p_1, p_2, \ldots, p_k$, such that $t > T$ and

$$| t\alpha_m - p_m - a_m | < \epsilon \qquad (m = 1, 2, \ldots, k) .$$

Definition 7. For $\delta > 0$ and real numbers $\lambda_1, \ldots, \lambda_m$ the set of real numbers τ satisfying the conditions $| \lambda_\nu \tau | \leqq \delta \pmod{2\pi}$, $\nu = 1, \ldots, m$ is called a <u>$(\delta ; \lambda)$-neighborhood of zero</u>. Here the notation "$|a| < \epsilon \pmod{b}$" means that there exists an integer k such that $| a - kb | < \epsilon$.

Theorem 7. For each almost periodic function f(t) and each positive number ϵ the set $\{\tau_f(\epsilon)\}$ of ϵ-translation numbers of f(t) contains a $(\delta;\lambda)$-neighborhood of zero.

Theorem 8. Each $(\delta;\lambda)$-neighborhood of zero is relatively dense.

For a thorough discussion of the main parts of the classical Bohr-theory of almost periodic functions, as well as an introduction to some of the generalizations to the class of Lebesgue measurable functions, the reader is referred to the 1932 book of A. S. Besicovitch [6]. This book also contains an introduction to the theory of analytic almost periodic functions of a complex variable.

B. Topological Spaces

This section contains the necessary topological background for Section IIC and Chapter III. Only a very few of the concepts given here are needed for Chapter IV. In fact, only a very special metric space is used in Chapter IV, namely, Euclidean N-space. It is assumed there that the reader is familiar with the rudiments of the theory of Euclidean N-dimensional vector spaces.

However, for Chapter III we shall need the notion of a general topological space. A <u>topology</u> for a set X is a family $\underline{T}$ of subsets of X which satisfies the following axioms:

(a) $X \in \underline{T}$ and $\emptyset \in \underline{T}$ (where $\emptyset$ denotes the null set).

(b) The intersection of any two members of $\underline{T}$ is a member of $\underline{T}$.

(c) The union of the members of each subfamily of $\underline{T}$ is a member of $\underline{T}$.

The set X is called the <u>space</u> of the topology $\underline{T}$, and $\underline{T}$ is called a <u>topology</u> for X. The pair $(X,\underline{T})$ is called a <u>general topological space</u>, or simply a <u>topological space</u>. The members of the family $\underline{T}$ are called the <u>open sets</u> of $(X,\underline{T})$. A set N in a topological space $(X,\underline{T})$ is called a <u>neighborhood</u> of a point x if N contains an open set containing x. The family of all neighborhoods of a point x is called the <u>neighborhood system</u> of x. A subset A of a topological space is said to be <u>closed</u> if its complement $\complement A$ is open. A point x is called a <u>limit point</u> of a subset A if every neighborhood of x contains at least one point of A other than x. A subset of a topological space is closed if and only if it contains all of its limit points. The intersection of all closed sets containing A is called the <u>closure</u> of A, and is denoted by $\overline{A}$. The closure of any set is the union of the set and the set of its limit points.

A family $\underline{B}$ of sets is said to form a <u>base</u> for a topology $\underline{T}$ if

(a) $\underline{B}$ is a subfamily of $\underline{T}$, and

(b) for each point x of the space and each neighborhood N of x, there is a member B of $\underline{B}$ such that $x \varepsilon B \subset N$.

A <u>base for the neighborhood system of a point</u> x, or a <u>local base</u> at x, is a family of neighborhoods of x such that every neighborhood of x contains a member of the family.

A topological space X is said to be <u>separable</u> if there is a finite or denumerable subset whose closure is X.

Sometimes it is convenient to consider a subset of a topological space as a topological space in its own right. Accordingly, if X' is a subset of a space $(X,\underline{T})$, then the family

$$\underline{T}' = \{U' : U' = U \cap X' \text{ for some } U \text{ in } \underline{T}\}$$

is called the <u>relative topology</u> for X', or the <u>relativization</u> of $\underline{T}$ to X'. The topological space $(X',\underline{T}')$ is called a subspace of the space $(X,\underline{T})$.

We shall also need the notion of a "product" of several (perhaps an infinite number) of topological spaces. This requires the concept of a subbase of a topological space. A family $\underline{S}$ of sets is called a <u>subbase</u> for a topology $\underline{T}$ if the family of all finite intersections of members of $\underline{S}$ is a base for $\underline{T}$. If $\{X_\alpha : \alpha \varepsilon A\}$ is a class of sets, their <u>Cartesian product</u> is the set $X = \times\{X_\alpha : \alpha \varepsilon A\}$ of all functions x defined on A such that for each α in A,

$x(\alpha) \in X_\alpha$. If each X_α is a topological space, then for each fixed α_0 in A, let U_{α_0} be an open subset of X_{α_0}, and, for $\alpha \neq \alpha_0$, let U_α denote X_α. A topology $\underline{T}$ for X can then be determined by the requirement that the class of all sets of the form $\times \{U_\alpha : \alpha \in A\}$ be a subbase for $\underline{T}$. This family of open sets for X is called the _product topology_ of the topologies in the sets X_α, α in A.

A function f on a topological space $(X,\underline{T})$ into a topological space $(X',\underline{T}')$ is said to be _continuous_ if the inverse image of each open set is open. f is said to be _continuous at a point_ x if the inverse image of each neighborhood of f(x) is a neighborhood of x. A topology $\underline{T}$ for a set X is called _completely regular_ if for each x in X and each neighborhood N of x there is a continuous function f on X to the closed unit interval such that f(x) = 0 and f is identically one on the complement of N.

A family $\underline{C}$ is called a _cover_ of a set B if B is a subset of the union $\bigcup \{C : C \in \underline{C}\}$. If each C in $\underline{C}$ is an open set the family $\underline{C}$ is called an _open cover_ of B. A subfamily of $\underline{C}$ which is also a cover of B is called a _subcover_ of $\underline{C}$. A topological space $(X,\underline{T})$ is said to be _compact_ if each open cover of X has a finite subcover.

We shall have occasion to use the following theorems regarding compact topological spaces.

Theorem 1. The Cartesian product of a collection of compact topological spaces is compact relative to the product topology.

Theorem 2. A closed subset of a compact space is compact.

Theorem 3. Let f be a continuous function on a compact topological space X onto a topological space Y. Then Y is compact.

A topology $\underline{T}$ for a set X is called <u>Hausdorff</u> if whenever x and y are distinct points of X there exist disjoint neighborhoods of x and y.

Theorem 4. A compact subset of a Hausdorff space is closed.

A function on the set of positive integers is called a <u>sequence</u>. Its value at the integer n is denoted by x_n. A point x of a topological space $(X,\underline{T})$ is called a <u>cluster point</u> of a sequence x_n if for each neighborhood N of x, x_n is in N for infinitely many distinct integers n. A point x of a topological space $(X,\underline{T})$ is called a <u>limit</u> of a sequence x_n if for each neighborhood N of x there exists an integer m(N) such that x_n is in N for all n greater than m(N). In this case x_n is said to <u>converge</u> to x. In general a sequence may have more than one limit. However, in Hausdorff spaces convergence of sequences is unique. That is, if $(X,\underline{T})$ is a Hausdorff space then each sequence in X converges to at most one point. Of course, even in a Hausdorff space, a given sequence can have more than one cluster point. A sequence y_n is called a <u>subsequence</u> of a sequence x_n if there is a sequence m_n of positive integers such that $y_n = x_{m_n}$ for each n, and for each integer m' there is an integer n' such that $m_n \geqq m'$ whenever $n \geqq n'$.

Theorem 5. Every sequence in a compact space has a cluster point.

Let A be a subset of a topological space $(X,\underline{T})$. If x_n is a sequence in A and if x_n converges to a point x not in A, then x is a limit point of A. However, the converse relationship does not necessarily hold in a general topological space. That is, in general, it is possible that a point x (not in A) can be a limit point of A and yet no sequence in A converges to x. This "inadequacy" of sequences for the description of limit points can be overcome only by a rather broad generalization of the concept of a sequence. For an exposition of an adequate theory of convergence in general topological spaces the reader is referred to the chapter on Moore-Smith convergence in J. L. Kelley's book [12], and to the references cited there. However, we shall not need this more general theory of convergence.

A <u>metric</u> for a set X is a function d on the Cartesian product $X \times X$ to the non-negative real numbers such that for all points x, y and z of X the following four axioms are fulfilled:

(a) $d(x,y) = d(y,x)$.

(b) $d(x,y) \leqq d(x,z) + d(z,y)$.

(c) If $x = y$, then $d(x,y) = 0$.

(d) If $d(x,y) = 0$, then $x = y$.

A function which satisfies (a), (b) and (c), but not necessarily (d) is called a <u>pseudometric</u>. If d is a metric for a set X then the family $\underline{B}$ of all "spheres" of the form

$$S(\epsilon ;x) = \{ y : d(x,y) < \epsilon \}$$

for x in X and positive ϵ, is a base for a topology for X. This topology is called the <u>metric topology generated by d</u>. The pair (X,d) with this topology is called a <u>metric space</u>. A topological space $(X,\underline{T})$ is said to be <u>metrizable</u> if there exists a metric d for X such that the topology $\underline{T}$ is identical with the metric topology generated by d. Every metric space is Hausdorff and completely regular. Furthermore, sequences provide an adequate theory of convergence in metric spaces. That is, a point x is a limit point of a subset A of a metric space (X,d) if and only if there exists a sequence x_n in $A - \{x\}$ such that x_n converges to x.

The following concepts for metric spaces will guide us in our construction of a notion of completeness for uniform spaces.

Theorem 6. A metric space is compact if and only if every sequence has a convergent subsequence.

Definition 1. A metric space (X,d) is called totally bounded if for each positive ϵ there exist a finite number of points $x_1,\ldots,x_{n(\epsilon)}$ of X such that

$$X = S(\epsilon;x_1)\cup\ldots\cup S(\epsilon;x_{n(\epsilon)}) .$$

A subset A of a metric space is called totally bounded if it is a totally bounded subspace.

Definition 2. A sequence x_n in a metric space (X,d) is called a Cauchy sequence if for each positive ϵ there exists an integer $k(\epsilon)$ such that

$$d(x_n,x_m) < \epsilon \quad \text{for all } n,\ m > k(\epsilon) .$$

Theorem 7. A metric space (X,d) is totally bounded if and only if each sequence in X has a Cauchy subsequence.

Definition 3. A metric space (X,d) is called complete if every Cauchy sequence in the space converges to a point of X.

Theorem 8. A closed subset of a complete metric space is complete.

Theorem 9. A Cauchy sequence in a metric space has at most one cluster point and converges to it.

C. Uniform Spaces

A uniform space can be described in terms of a family of "relations" which satisfies certain axioms. A subset R of the Cartesian product $X \times X$ is called a <u>binary relation</u> in X, or simply a <u>relation</u>. The <u>domain</u> of a relation R is the set of all first coordinates of members of R, and its <u>range</u> is the set of all second coordinates. The <u>inverse</u> of a relation R is denoted by R^{-1} and is defined by the equation

$$R^{-1} = \{(x,y) : (y,x) \in R\}.$$

If R and S are relations their <u>composition</u> $R \circ S$ is defined by the equation

$$R \circ S = \{(x,y) : \text{for some } z,\ (x,z) \in S \text{ and } (z,y) \in R\}.$$

The <u>identity</u> relation (or the <u>diagonal</u>) is denoted by Δ, or $\Delta(X)$, and is defined by the equation

$$\Delta(X) = \{(x,x) : x \in X\}.$$

If R is a relation in X and A is a subset of X, then $R[A]$ denotes the set of all <u>R-relatives</u> of points of A, and is defined by the equation

$$R[A] = \{y : (x,y) \in R \text{ for some } x \in A\}.$$

A function is a relation f such that if $(x,y) \in f$ and $(x,z) \in f$, then $y = z$. A relation R is called <u>symmetric</u> if $R = R^{-1}$. Arbitrary relations R, S and T satisfy the following formulas:

(a) $(R^{-1})^{-1} = R$ and $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

(b) $R \circ (S \circ T) = (R \circ S) \circ T$ and $(R \circ S)[A] = R[S[A]]$.

(c) $R[\bigcup\{X_\alpha : \alpha \in A\}] = \bigcup\{R[X_\alpha] : \alpha \in A\}$.

A <u>uniformity</u> for a set X can now be described as a non-empty family $\underline{U}$ of relations in X such that the following axioms are satisfied:

(a) Each member of $\underline{U}$ contains the diagonal $\Delta(X)$.

(b) If $U \in \underline{U}$, then $U^{-1} \in \underline{U}$.

(c) If $U \in \underline{U}$, then $V \circ V \subset U$ for some $V \in \underline{U}$.

(d) If U and V are members of $\underline{U}$, then $U \cap V$ is a member of $\underline{U}$.

(e) If $U \in \underline{U}$ and $U \subset V \subset X \times X$, then $V \in \underline{U}$.

The pair $(X,\underline{U})$ is called a uniform space. A subfamily $\underline{B}$ of a uniformity $\underline{U}$ is called a <u>base</u> for $\underline{U}$ if each member of $\underline{U}$ contains a member of $\underline{B}$. A subfamily $\underline{S}$ of $\underline{U}$ is said to be a <u>subbase</u> for $\underline{U}$ if the family of all finite intersections of members of $\underline{S}$ is a base for $\underline{U}$.

If $(X,\underline{U})$ is a uniform space, then the family $\underline{T}$ defined as

$$\{V : V \subset X, \text{ and for each } x \in V,\ U[x] \subset V \text{ for some } U \text{ in } \underline{U}\}$$

is a topology for X. This topology is called the <u>uniform topology</u> of the uniform space $(X,\underline{U})$.

Theorem 1. If $\underline{B}$ is a base (subbase) for a uniformity $\underline{U}$, then for each x in X the family of sets $U[x]$ for U in $\underline{B}$ is a base (subbase, respectively) for the neighborhood system of x.

In fact, the family of all sets $U[x]$ for U in $\underline{U}$ is actually identical with the neighborhood system of x (for the uniform topology). It is of interest to note that a topology $\underline{T}$ for a set X is the uniform topology for some uniformity for X if and only if the topological space $(X,\underline{T})$ is completely regular. The uniform topology of a uniform space is Hausdorff if and only if each point is a closed set. Hence the uniform topology is Hausdorff if and only if $\bigcap \{ U : U \in \underline{U} \}$ is the diagonal $\Delta(X)$. A uniform space whose uniform topology is Hausdorff is called a <u>Hausdorff uniform space</u>.

If $(X,\underline{U})$ is a uniform space and X' is a subset of X, then the family $\underline{U}'$ of all intersections of the members of $\underline{U}$ with $X' \times X'$ is a uniformity for X'. This uniformity is called the <u>relativization</u> of $\underline{U}$ to X', or the <u>relative uniformity</u> for X'. $(X',\underline{U}')$ is called a <u>subspace</u> of $(X,\underline{U})$. The uniform topology of the relative uniformity $\underline{U}'$ is the relativized uniform topology of $\underline{U}$.

If $(X_\alpha,\underline{U}_\alpha)$ is a uniform space for each member α of an index set A, then the family of all sets of the form $\{(x,y) : (x_\alpha,y_\alpha) \in U\}$ for α in A and U in $\underline{U}_\alpha$ is a subbase for a uniformity $\underline{U}$ for the Cartesian product set $\times\{X_\alpha : \alpha \in A\}$. This uniformity $\underline{U}$ is called the <u>product uniformity</u> of the uniformities $\underline{U}_\alpha$, $\alpha \in A$. The uniform topology of the product uniformity is the product topology of the uniform topologies.

As an example of a uniform space let us consider a metric space (X,d). Let $G(\epsilon)$ denote the subset of $X \times X$ defined as

$$\{(x,y) : d(x,y) < \epsilon\}.$$

Let

$$\underline{U} = \{U : U \subset X \times X \text{ and } G(\epsilon) \subset U \text{ for some } \epsilon > 0\}.$$

Then $\underline{U}$ is a uniformity for the metric space (X,d) and its uniform topology is identical to the metric topology of (X,d). In particular the space of real numbers and the space of complex numbers are uniform spaces whose uniformities are generated in the above manner by the familiar metric $d(x,y) = |x - y|$.

Definition 1. If f is a function on a uniform space $(X,\underline{U})$ with values in a uniform space $(Y,\underline{V})$, then f is said to be <u>uniformly continuous</u> if for each V in $\underline{V}$ there exists a U in $\underline{U}$ such that

$$(f(x), f(y)) \in V \text{ whenever } (x,y) \in U .$$

This reduces to the usual definition when $(X,\underline{U})$ and $(Y,\underline{V})$ are metric spaces. Recall that a pseudo-metric d has all the properties of a metric except that $d(x,y)$ can equal zero without $x = y$. Let $(X,\underline{U})$ be a uniform space and let d be a pseudo-metric for X. Then d is uniformly continuous on $X \times X$ (relative to the product uniformity) if and only if the set

$$G_{d,\epsilon} = \{(x,y) : d(x,y) < \epsilon\}$$

is a member of $\underline{U}$ for each positive number ϵ. Let $\underline{P}$ be a family of pseudo-metrics d for a set X. Then

$$\{G_{d,\epsilon} : d \in \underline{P} \text{ and } \epsilon > 0\}$$

is a subbase for a uniformity $\underline{U}$ for X. This uniformity is said to be the <u>uniformity generated by $\underline{P}$</u>. A uniform space $(X,\underline{U})$ is called metrizable if there is a single metric d such that $\underline{U}$ is the uniformity generated by $\underline{P} = \{d\}$. A necessary and sufficient condition that a uniform space $(X,\underline{U})$ be metrizable is that $\underline{U}$ have a countable base and its uniform topology be Hausdorff. Each uniformity for X is generated by the family of all pseudo-metrics which are uniformly continuous on $X \times X$. Let $(X,\underline{U})$ be a given uniform space. Then the family $\underline{G}$, of all pseudo-metrics d which are uniformly continuous on $X \times X$, is called the <u>gage</u> of the uniformity $\underline{U}$.

Theorem 2. Each continuous function on a compact uniform space to a uniform space is uniformly continuous.

Definition 2. A uniform space $(X,\underline{U})$ is called <u>totally bounded</u> if for each $U \in \underline{U}$ there is a finite subset $A = \{x_1,\ldots,x_n\}$ of X such that $U[A] = X$. A subset X' of a uniform space is called <u>totally bounded</u> if X' is a totally bounded subspace.

Theorem 3. Let $(X,\underline{U})$ be a uniform space and let $\underline{G}$ be the gage of $\underline{U}$. Then:

(a) The family of all sets of the form

$$G_{d,\epsilon} = \{(x,y) : d(x,y) < \epsilon\}$$

for d in $\underline{G}$ and ϵ positive is a base for the uniformity $\underline{U}$. $G_{d,\epsilon}$ is an open set in the product space $X \times X$.

(b) The family of all sets of the form

$$F_{d,\epsilon} = \{(x,y) : d(x,y) \leqq \epsilon\}$$

for d in $\underline{G}$ and ϵ positive is a base for the uniformity $\underline{U}$. $F_{d,\epsilon}$ is a closed set in the product space $X \times X$.

(c) A function f on X to a uniform space $(Y,\underline{V})$ with gage $\underline{H}$ is uniformly continuous if and only if for each p in $\underline{H}$ and each positive ϵ there is a d in $\underline{G}$ and a positive δ such that

$$p(f(x), f(y)) < \epsilon \text{ whenever } d(x,y) < \delta .$$

(d) $(X,\underline{U})$ is Hausdorff if and only if for each pair of distinct points x_1 and x_2 there exists a pseudo-metric d' in $\underline{G}$ such that $d'(x_1, x_2) \neq 0$.

We stated before that a uniform space $(X,\underline{U})$ is Hausdorff if and only if

$$\bigcap \{U : U \in \underline{U}\} = \Delta(X) .$$

The statement in (d) of Theorem 3 is easily seen to be a consequence of this condition. To this end we first note that if $\underline{B}$ is a base for $\underline{U}$, then since each U in $\underline{U}$ contains a V in $\underline{B}$

$$\bigcap\{U : U \in \underline{U}\} = \bigcap\{V : V \in \underline{B}\} .$$

Now consider the base $\{G_{d,\epsilon} : d \in \underline{G}, \epsilon > 0\}$. Then the above condition can be stated as follows: A uniform space (X,U) is Hausdorff if and only if

$$\bigcap\{G_{d,\epsilon} : d \in \underline{G}, \epsilon > 0\} = \Delta(X) .$$

Now suppose that $(X,\underline{U})$ is Hausdorff but that for some pair of distinct points (x_1, x_2), $d(x_1, x_2) = 0$ for all d in the gage $\underline{G}$. Then $(x_1, x_2) \in G_{d,\epsilon}$ for every d in $\underline{G}$ and positive ϵ, and hence

$$(x_1, x_2) \in \bigcap\{G_{d,\epsilon} : d \in \underline{G}, \epsilon > 0\}$$

contrary to hypothesis. On the other hand, suppose that for each pair of distinct points (x_1, x_2) there exists a d' in $\underline{G}$ such that $d(x_1, x_2) \neq 0$. Then

$$(x_1, x_2) \notin G_{d', 1/2\, d'(x_1, x_2)} ,$$

and hence $\bigcap\{G_{d,\epsilon} : d \in \underline{G}, \epsilon > 0\}$ cannot contain the pair (x_1, x_2) when $x_1 \neq x_2$. Consequently

$$\bigcap\{G_{d,\epsilon} : d \in \underline{G}, \epsilon > 0\} = \Delta(X)$$

and (X,$\underline{U}$) is Hausdorff.

This completes the preliminary information on uniform spaces. In Section IIIB a concept of completeness for uniform spaces will be needed. Accordingly, such a concept, sufficient for our purposes, will be developed there. The usual concept of a complete uniform space, defined in terms of Cauchy nets ("nets" being generalized sequences), is stronger than necessary for the generalization of Tornehave's theory given in Chapter III of this thesis.

III. ALMOST PERIODIC MOVEMENTS IN UNIFORM SPACES

This part of the thesis indicates how H. Tornehave's theory of almost periodic movements in metric spaces [11] can be generalized to a theory of almost periodic movements in uniform spaces. Only those parts of the generalization of Tornehave's theory which are needed to establish the "approximation theorem" have been given here. Throughout this part we shall be dealing with functions of a real variable t with values in a uniform space $(X,\underline{U})$.

A. Uniformly Continuous Families of Almost Periodic Movements

Definition 1. Let $(X,\underline{U})$ be a uniform space. A continuous function $x = f(t)$, $-\infty < t < \infty$, $x \in X$, is called a <u>movement</u> in X.

Definition 2. Let U be a member of the uniformity $\underline{U}$. If a real number $\tau = \tau_f(U)$ satisfies the condition

$$(f(t), f(t+\tau)) \in U \text{ for all real } t,$$

then τ is called a <u>U-translation number</u> of $f(t)$. The set of all U-translation numbers of a function $f(t)$ is denoted by $\{\tau_f(U)\}$.

Definition 3. Let $\underline{G}$ be the gage of $\underline{U}$, $d \in \underline{G}$ and ϵ a positive number. If a real number $\tau = \tau_f(\epsilon;d)$ satisfies the condition

$$d(f(t), f(t+\tau)) \leqq \epsilon \quad \text{for all real } t,$$

then τ is called an <u>$(\epsilon;d)$-translation number</u> of $f(t)$. The set of all $(\epsilon;d)$-translation numbers of a function $f(t)$ is denoted by $\{\tau_f(\epsilon;d)\}$.

Since by Theorem 3 (b) of Section IIC the sets of the form $F_{d,\epsilon} = \{(x,y) : d(x,y) \leqq \epsilon\}$ form a base for $\underline{U}$, it follows that

$$\tau_f(F_{d,\epsilon}) = \tau_f(\epsilon;d) .$$

Definition 4. A movement $x = f(t)$ is called <u>almost periodic</u> if the set $\{\tau_f(U)\}$ of U-translation numbers is relatively dense for each U in $\underline{U}$.

Theorem 1. A movement $x = f(t)$ is almost periodic if and only if the set $\{\tau_f(\epsilon;d)\}$ of $(\epsilon;d)$-translation numbers is relatively dense for each d in $\underline{G}$ and positive ϵ.

Proof. This theorem is an immediate consequence of the following three facts:

(a) $\{F_{d,\epsilon} : d \in \underline{G} \text{ and } \epsilon > 0\}$ is a base for $\underline{U}$.

(b) $F \subset U$ implies $\{\tau_f(F)\} \subset \{\tau_f(U)\}$.

(c) A set of real numbers which contains a relatively dense subset is relatively dense.

Definition 5. If $f(t)$ is an almost periodic movement then the sets $\{\mathcal{T}_f(U)\}$ and $\{\mathcal{T}_f(\epsilon;d)\}$ are relatively dense. That is, there exists a number l_U such that every interval of length l_U contains at least one number of $\{\mathcal{T}_f(U)\}$, and there exists a number $l_{\epsilon;d}$ such that every interval of length $l_{\epsilon;d}$ contains at least one number of $\{\mathcal{T}_f(\epsilon;d)\}$. Each such number l_U is called a <u>U-inclusion interval</u> for $f(t)$, and each such number $l_{\epsilon;d}$ is called an <u>$(\epsilon;d)$-inclusion interval</u> for $f(t)$.

Definition 6. Let R denote a compact separable metric space consisting of points denoted $v, v_1, v_2, \ldots$ and with neighborhoods $N(v)$. A function $x = f(t;v)$, defined for all real t and all v in R, with values in X is called a <u>uniformly continuous family of almost periodic movements</u> if

(a) the function $f(t;v)$ is almost periodic in t for each v in R, and if

(b) to each U in $\underline{U}$ and v_o in R there corresponds a neighborhood $N_U(v_o)$ such that $(f(t;v_o), f(t;v)) \in U$ for all real t and all v in $N_U(v_o)$.

Definition 7. A set $\{f(t)\}$ of almost periodic movements in X is called a <u>uniformity set</u> if for each d in $\underline{G}$ there exists a real-valued almost periodic function $g_d(t)$ such that the set of common $(\epsilon;d)$-translation numbers of all the functions $f(t)$ of the set contains the set of ϵ-translation numbers of $g_d(t)$.

Theorem 2. For each d in $\underline{G}$ the d-diameter of the range of an almost periodic movement f(t) is finite.

Proof. Let $d \varepsilon \underline{G}$. The real-valued function $\phi_d(t) = d(f(o), f(t))$ satisfies the condition

$$|\phi_d(t_2) - \phi_d(t_1)| = |d(f(o), f(t_2)) - d(f(o), f(t_1))| \leqq d(f(t_1), f(t_2)) .$$

This implies that $\phi_d(t)$ is an almost periodic function. Hence we may choose a real number K_d such that

$$\phi_d(t) = d(f(o), f(t)) \leqq \frac{K_d}{2} \text{ for all real } t.$$

But then

$$d(f(t_1), f(t_2)) \leqq d(f(o), f(t_1)) + d(f(o), f(t_2)) \leqq K_d$$

for all real t_1 and t_2. This completes the proof.

Theorem 3. An almost periodic movement $x = f(t)$ is uniformly continuous.

Proof. It must be shown that for each U in $\underline{U}$ there exists a positive number δ such that $(f(t_1), f(t_2)) \varepsilon U$ for all t_1 and t_2 satisfying $|t_1 - t_2| < \delta$. Let $U \varepsilon \underline{U}$ be given. Since the family of all sets $F_{d,\epsilon}$ for d in $\underline{G}$ and ϵ positive is a base for $\underline{U}$ there exists a d in $\underline{G}$ and a positive ϵ such that $F_{d,\epsilon} \subset U$. Furthermore, since each finite closed interval of real numbers is compact and since by Theorem 2 of Section IIC each continuous function on a compact uniform space to another uniform space is uniformly continuous, it follows that $f(t)$ is uniformly continuous on every finite interval. Therefore there exists a δ, $0 < \delta < 1$, such that

$$d(f(t_1), f(t_2)) \leqq \frac{\epsilon}{3} \tag{1}$$

for all t_1 and t_2 in the interval $(o, 1_{\epsilon/3;d} + 1)$ provided only that $|t_1 - t_2| < \delta$, where $1_{\epsilon/3;d}$ is an $(\epsilon/3;d)$-inclusion interval for $f(t)$. Let t' and t'' be any two real numbers such that

$$|t' - t''| < \delta .$$

Suppose $t' < t''$. Then since $f(t)$ is almost periodic there must exist at least one τ in $\{\tau_f(\epsilon/3;d)\}$ such that

$$-t' < \tau < 1_{\epsilon/3;d} - t' .$$

From $-t'' < -t' < \tau < 1_{\epsilon/3;d} - t'$ and $o < t'' - t' < \delta < 1$ it follows that

$$o < \tau + t'' < 1_{\epsilon/3;d} + t'' - t' < 1_{\epsilon/3;d} + 1$$

and

$$o < \tau + t' < 1_{\epsilon/3;d} < 1_{\epsilon/3;d} + 1 .$$

That is, $\tau + t''$ and $\tau + t'$ both belong to the open interval $(o, 1_{\epsilon/3;d} + 1)$, and since $|(t' + \tau) - (t'' + \tau)| < \delta$, it follows from (1) that

$$d(f(t' + \tau), f(t'' + \tau)) \leqq \epsilon/3 .$$

Consequently we have

$$d(f(t'), f(t'')) \leqq d(f(t'), f(t' + \tau)) + d(f(t' + \tau), f(t'' + \tau)) + d(f(t'' + \tau), f(t'')) \leqq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon .$$

That is,

$$(f(t'), f(t'')) \in F_{d,\epsilon} \subset U \text{ whenever } |t' - t''| < \delta .$$

This completes the proof of Theorem 3.

Theorem 4. Let $x = f(t;v)$, $-\infty < t < \infty$, $v \in R$, $x \in X$, be a uniformly continuous family of almost periodic movements. Then for each d in $\underline{G}$ there exists a constant K_d such that

$$d(f(t_1;v), f(t_2;v)) \leqq K_d$$

for all real values of t_1 and t_2 and for all v in R.

Proof. Let d be an arbitrary pseudo-metric of the gage $\underline{G}$. Let v_o be an arbitrary point of the space R. Then it follows from condition (b) of Definition 6 that for each positive ϵ there exists a neighborhood $N_{F_{d,\epsilon}}(v_o)$ such that

$$d(f(t;v_o), f(t;v)) \leqq \epsilon \tag{2}$$

for all v in $N_{F_{d,\epsilon}}(v_o)$ and for all real t. Also from Theorem 2 there exists a $K_d(v_o)$ such that

$$d(f(t_1;v_o),\ f(t_2;v_o)) \leqq K_d(v_o) \tag{3}$$

for all real t_1 and t_2. Hence from (2) and (3)

$$\begin{aligned} d(f(t_1;v),\ f(t_2;v)) \leqq\ & d(f(t_1;v),\ f(t_1;v_o)) \\ & + d(f(t_1;v_o),\ f(t_2;v_o)) \\ & + d(f(t_2;v_o),\ f(t_2;v)) \leqq \epsilon + K_d(v_o) + \epsilon\ , \end{aligned}$$

for all v in $N_{F_{d,\epsilon}}(v_o)$ and all real t_1 and t_2. But the family $\{ N_{G_{d,\epsilon}}(v) : v \in R \}$ is an open covering of the compact space R, and $G_{d,\epsilon} \subset F_{d,\epsilon}$ for every d in $\underline{G}$ and positive ϵ. Hence we can find a finite number of points $v_1,\ldots,v_n$ of R such that R is covered by $\{ N_{F_{d,\epsilon}}(v_\nu) : \nu = 1,\ldots,n \}$. Then

$$d(f(t_1;v),\ f(t_2;v)) \leqq 2\epsilon + \max_{\nu=1,\ldots,n} K_d(v_\nu) = K_d$$

for every v in R and for all real t_1 and t_2. This completes the proof.

Definition 8. For each d in $\underline{G}$, the function

$$e_d(\tau;v) = \sup_{-\infty<t<\infty} d(f(t;v),\ f(t+\tau;v))$$

is called the <u>d-translation</u> <u>function</u> of the uniformly continuous family of almost period movements $f(t;v)$, and

$$e_d(\tau) = \sup_{v \in R} e_d(\tau;v)$$

is called the <u>d-translation</u> <u>majorant</u> of $f(t;v)$.

Theorem 5. The d-translation function $e_d(\tau;v)$ and the d-translation majorant $e_d(\tau)$ are real, non-negative, and bounded.

Proof. The boundedness follows immediately from Theorem 4, while the other statements are obvious from the definition of a pseudo-metric.

Theorem 6. The d-translation function $e_d(\tau;v)$ and the d-translation majorant $e_d(\tau)$ satisfy the conditions

(a) $e_d(o;v) = e_d(o) = o$

(b) $e_d(\tau;v) = e_d(-\tau;v)$

(c) $e_d(\tau) = e_d(-\tau)$

(d) $e_d(\tau_1 + \tau_2;v) \leqq e_d(\tau_1;v) + e_d(\tau_2;v)$

(e) $e_d(\tau_1 + \tau_2) \leqq e_d(\tau_1) + e_d(\tau_2)$.

Proof. (a), (b) and (c) follow immediately from Definition 8. For (d) we have

$$e_d(\tau_1 + \tau_2;v) = \sup_{-\infty < t < \infty} d(f(t;v), f(t + \tau_1 + \tau_2;v))$$

$$\leqq \sup_{-\infty < t < \infty} d(f(t;v), f(t + \tau_1;v))$$

$$+ \sup_{-\infty < t < \infty} d(f(t + \tau_1;v), f(t + \tau_1 + \tau_2;v))$$

$$= e_d(\tau_1;v) + e_d(\tau_2;v) .$$

For (e) we have

$$e_d(\tau_1 + \tau_2) = \sup_{v \varepsilon R} e_d(\tau_1 + \tau_2;v)$$

$$\leqq \sup_{v \varepsilon R} e_d(\tau_1;v) + \sup_{v \varepsilon R} e_d(\tau_2;v)$$

$$= e_d(\tau_1) + e_d(\tau_2) .$$

Theorem 7. Let v be a given point of R and d a given pseudo-metric of $\underline{G}$. Then the d-translation function $e_d(t;v)$ is an almost periodic function of t, and

$$\{\tau_f(\epsilon;d)\} = \{\tau_e(\epsilon)\} = \{\tau : e_d(\tau;v) \leqq \epsilon\}.$$

Proof. A number τ is an $(\epsilon;d)$-translation number of $f(t;v)$ if and only if

$$e_d(\tau;v) = \sup_{-\infty<t<\infty} d(f(t;v), f(t+\tau;v)) \leqq \epsilon .$$

Therefore $\{\tau_f(\epsilon;d)\} = \{\tau : e_d(\tau;v) \leqq \epsilon\}$. On the other hand we have by Theorem 6

$$e_d(t+\tau;v) - e_d(t;v) \leqq e_d(\tau;v)$$

$$e_d(t;v) - e_d(t+\tau;v) \leqq e_d(-\tau;v) = e_d(\tau;v),$$

and hence $|e_d(t+\tau;v) - e_d(t;v)| \leqq e_d(\tau;v)$. But $|e_d(o+\tau;v) - e_d(o;v)| = e_d(\tau;v)$. Therefore,

$$\sup_{-\infty<t<\infty} |e_d(t+\tau;v) - e_d(t;v)| = e_d(\tau;v).$$

It follows that τ is an ϵ-translation number for $e_d(t;v)$ if and only if $e_d(\tau;v) \leqq \epsilon$. Therefore, $\{\tau_e(\epsilon)\} = \{\tau : e_d(\tau;v) \leqq \epsilon\} = \{\tau_f(\epsilon;d)\}$. But $\{\tau_f(\epsilon;d)\}$ is relatively dense. Furthermore, from Theorem 3 it follows that this set of translation numbers contains an interval about the origin. Hence $e_d(t;v)$ is continuous and is therefore an almost periodic function.

Theorem 8. For each d in G the d-translation majorant $e_d(t)$ is an almost periodic function.

Proof. From Theorem 6 it follows that

$$e_d(t+\tau) - e_d(t) \leqq e_d(\tau)$$

$$e_d(t) - e_d(t+\tau) \leqq e_d(-\tau) = e_d(\tau),$$

hence $|e_d(t+\tau) - e_d(t)| \leqq e_d(\tau)$. But $|e_d(o+\tau) - e_d(o)| = e_d(\tau)$. Hence

$$e_d(\tau) = \sup_{-\infty < t < \infty} |e_d(t+\tau) - e_d(t)| . \qquad (4)$$

Therefore τ is an ϵ-translation number of $e_d(t)$ if and only if $e_d(\tau) \leqq \epsilon$, that is, if and only if $e_d(\tau;v) \leqq \epsilon$ for all v in R. Let v be an arbitrary point of R. By condition (b) of Definition 6 there exists a neighborhood $N_{F_{d,\epsilon/3}}(v)$ of v such that

$$d(f(t;v'), f(t;v)) \leqq \epsilon/3 \qquad (5)$$

for all real t and all v' in $N_{F_{d,\epsilon/3}}(v)$. Since R is compact we may choose a finite number of points $v_1,\dots,v_n$ such that

$$R \subset N_{F_{d,\epsilon/3}}(v_1) \cup \dots \cup N_{F_{d,\epsilon/3}}(v_n) .$$

The sum $e_d(t;v_1) + \dots + e_d(t;v_n) = E_d(t)$ is an almost periodic function for each d in G. If v is an arbitrary point of R we can choose ν, $1 \leqq \nu \leqq n$, such that $v \in N_{F_{d,\epsilon/3}}(v_\nu)$. Now

$$e_d(\tau;v) = \sup_{-\infty < t < \infty} d(f(t;v),\, f(t+\tau;v))$$

$$\leqq \sup_{-\infty < t < \infty} \{ d(f(t;v),\, f(t;v_\nu))$$

$$+ d(f(t;v_\nu),\, f(t+\tau;v_\nu))$$

$$+ d(f(t+\tau;v_\nu),\, f(t+\tau;v)) \} ,$$

and thus from (5) it follows that

$$e_d(\tau;v) \leqq \sup_{-\infty < t < \infty} \{ \epsilon/3 + d(f(t;v_\nu),\, f(t+\tau;v_\nu))$$

$$+ \epsilon/3 \} \leqq 2/3\,\epsilon + e_d(\tau;v_\nu)$$

$$\leqq 2/3\,\epsilon + E_d(\tau) ,$$

for all v in R. We have thus proved that $e_d(\tau) \leqq \epsilon$ if $E_d(\tau) \leqq \epsilon/3$. But since

$$| E_d(t+\tau) - E_d(t) | \leqq E_d(\tau)$$

and

$$| E_d(o+\tau) - E_d(o) | = E_d(\tau) ,$$

it follows that

$$\sup_{-\infty < t < \infty} | E_d(t+\tau) - E_d(t) | = E_d(\tau) .$$

Hence $e_d(\tau;v) \leqq \epsilon$ for all v in R whenever τ is an $\epsilon/3$-translation number of the almost periodic function $E_d(t)$. This set of translation numbers is relatively dense and contains an interval about zero. Hence $e_d(t)$ is an almost periodic function.

Theorem 9. The set of all ϵ-translation numbers of the d-translation majorant $e_d(t)$ is identical to the set of common $(\epsilon;d)$-translation numbers of the almost periodic functions of the family $\{f(t;v) : v \varepsilon R\}$.

Proof. According to Theorem 7

$$\{\tau_f(\epsilon;d)\} = \{\tau : e_d(\tau, v) \leqq \epsilon\} .$$

Therefore the set of common $(\epsilon;d)$-translation numbers of the family $\{f(t;v) : v \varepsilon R\}$ is identical to the set $\{\tau : e_d(\tau;v) \leqq \epsilon$ for all $v \varepsilon R\}$. But this last set is the same as the set $\{\tau : e_d(\tau) \leqq \epsilon\}$, which in turn by (4) is the set of ϵ-translation numbers of $e_d(t)$.

Theorem 10. The set of almost periodic functions belonging to a uniformly continuous family of almost periodic movements is a uniformity set.

Proof. This theorem is an immediate consequence of Theorems 8 and 9 and Definition 7.

Theorem 11. The set of common U-translation numbers of all almost periodic movements belonging to a uniformly continuous family of almost period movements contains a $(\delta;\lambda)$-neighborhood of zero.

Proof. Let $U \varepsilon \underline{U}$ be given. Then there exists an $F_{d,\epsilon} \subset U$. According to Theorem 9 the set of common $(\epsilon;d)$-translation numbers of the family $f(t;v)$ is identical to the set $\{\tau_e(\epsilon)\}$ and by Theorem 7 of Section IIA this set contains a $(\delta;\lambda)$-neighborhood of zero. But $F_{d,\epsilon} \subset U$ implies $\{T_f(\epsilon;d)\} \subset \{T_f(U)\}$. This completes the proof.

* Theorem 12. The range of an almost periodic movement is totally bounded.

Proof. Let $U \in \underline{U}$ be given. Then there exists an $F_{d,\epsilon} \subset U$. Let $l = l_{\epsilon/2;d}$ be chosen so that every interval of length l contains an $(\epsilon/2;d)$-translation number of $f(t)$. According to Theorem 3 we can choose a positive δ such that $d(f(t_2), f(t_1)) \leqq \epsilon/2$ whenever $|t_1 - t_2| < \delta$. Let $x_\nu = f(\nu\delta)$, $\nu = 1, 2, \ldots, n$, where $n = [l/\delta]$ (the greatest integer in l/δ). Let x' be an arbitrary element of the range of $f(t)$. Then there exists a real number t' such that $x' = f(t')$, and there exists an $(\epsilon/2;d)$-translation number τ such that $-t' < \tau < l - t'$. Let t'' denote $t' + \tau$. Then $o < t'' < l$, and therefore an integer ν, $1 \leqq \nu \leqq n$, can be chosen such that $|t'' - \nu\delta| < \delta$. Consequently,

$$\begin{aligned} d(x', x_\nu) &= d(f(t'), f(\nu\delta)) \\ &\leqq d(f(t'), f(t'')) + d(f(t''), f(\nu\delta)) \\ &= d(f(t'), f(t'+\tau)) + d(f(t''), f(\nu\delta)) \\ &\leqq \epsilon/2 + \epsilon/2 = \epsilon . \end{aligned}$$

Hence every point x' of the range of $f(t)$ is contained in

$$\begin{aligned} F_{d,\epsilon}[x_1] \cup \cdots \cup F_{d,\epsilon}[x_n] &= F_{d,\epsilon}[\{x_1,\ldots,x_n\}] \\ &\subset U[\{x_1,\ldots,x_n\}] . \end{aligned}$$

Hence, by Definition 2 of Section IIC, the range of $f(t)$ is a totally bounded subspace of X. This completes the proof of Theorem 12.

B. Uniformly Continuous Families of Almost Periodic Movements in Complete Uniform Spaces

The concepts and theorems of this section and the next require a notion of completeness for a uniform space. The usual notion of completeness for uniform spaces is more stringent than is necessary for our purposes. For six distinct notions of completeness and their mutual relationships in convex topological linear spaces the reader is referred to a paper by G. W. Mackey, [9]. The following theorem for metric spaces provides a condition which we shall adopt as a definition for completeness in uniform spaces.

Theorem 1. A metric space is complete if and only if every closed and totally bounded subset is compact.

Proof. Suppose first that every closed and totally bounded subset of a metric space (X,d) is compact. Let x_n be a Cauchy sequence in X. Then the set A defined as

$$\{ y : y = x_n \text{ for some integer } n \}$$

is totally bounded. For let a positive number ϵ be given. Then since x_n is Cauchy there exists a positive integer

$N = N(\epsilon)$ such that $d(x_n, x_m) < \epsilon/2$ for all m and n greater than N. Therefore

$$A \subset S(\epsilon/2; x_1) \cup \ldots \cup S(\epsilon/2; x_{N+1}),$$

and hence by Definition 1 of Section IIB A is a totally bounded subspace of X.

Furthermore $\bar{A}$ (the closure of A) is totally bounded. For suppose $x \in \bar{A}$. Then there exists an integer k such that $x_k \in A$ and $d(x, x_k) < \epsilon/2$ and therefore

$$\begin{aligned} d(x, x_l) &\leqq d(x, x_k) + d(x_k, x_l) \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

where

$$l = \begin{cases} k & \text{if } k \leqq N \\ N+1 & \text{if } k > N \end{cases}.$$

Thus

$$\bar{A} \subset S(\epsilon; x_1) \cup \ldots \cup S(\epsilon; x_{N+1}).$$

Since $\bar{A}$ is closed and totally bounded it follows from our hypothesis that $\bar{A}$ is compact. By Theorem 5 of Section IIB, x_n has a cluster point x' in $\bar{A}$. By Theorem 9 of Section IIB, x_n converges uniquely to x'. Since x_n was an arbitrary Cauchy sequence in X, it follows that (X,d) is a complete metric space.

Conversely, suppose that every Cauchy sequence in (X,d) converges. That is, suppose that (X,d) is complete. Let A be a closed and totally bounded subset of X. Then we must show that A is compact. Since A is totally bounded it follows from Theorem 7 of Section IIB that every sequence in A has a Cauchy subsequence. Since A is closed it follows from Theorem 8 of Section IIB and the latter conclusion that every sequence in A has a convergent subsequence. But then by Theorem 6 of Section IIB, A is compact.

Definition 1. A uniform space will be called complete if and only if every closed and totally bounded subset is compact.

For the usual (stronger) notion of completeness in uniform spaces, in terms of Cauchy directed systems, the reader is referred to the Chapter on Moore-Smith convergence in John L. Kelley's book [12].

Theorem 2. The closure of a totally bounded subset of a uniform space is totally bounded.

Proof. Let A be a totally bounded subset of a uniform space $(X, \underline{U})$. Let $\bar{A}$ denote the closure of A. For each pseudo-metric d in the gage $\underline{U}$ and for each positive ϵ there exist a finite number of elements $x_1,\ldots,x_n$ of A such that

$$A \subset G_{d,\,\epsilon/2}[x_1] \bigcup \cdots \bigcup G_{d,\,\epsilon/2}[x_n] \ .$$

Let $x \in \bar{A}$. Then there exists an element x' in A such that

$d(x,x') < \epsilon/2$, and there exists an integer $1 \leqq \nu(x') \leqq n$ such that $d(x', x_{\nu(x')}) < \epsilon/2$. Hence

$$d(x,x_{\nu(x')}) \leqq d(x,x') + d(x',x_{\nu(x')})$$
$$< \epsilon/2 + \epsilon/2 = \epsilon .$$

Therefore each point of $\bar{A}$ is contained within a set of the form $G_{d,\epsilon}[x]$ for some ν, $1 \leqq \nu \leqq n$. That is,

$$\bar{A} \subset G_{d,\epsilon}[x_1] \cup \cdots \cup G_{d,\epsilon}[x_n] .$$

Hence $\bar{A}$ is totally bounded.

Theorem 3. A uniform space is compact if and only if it is totally bounded and complete.

Proof. Suppose first that $(X,\underline{U})$ is compact. Then since every closed subset is compact (by Theorem 2 of Section IIB), it follows that every closed and totally bounded subset is compact. That is, $(X,\underline{U})$ is complete. Furthermore, for each pseudo-metric d in the gage $\underline{G}$ and for each positive number ϵ, the family

$$\{ G_{d,\epsilon}[x] : x \varepsilon X \}$$

is an open cover of X. But since X is compact this cover must have a finite subcover. That is, there must exist a finite number of points $x_1,\ldots,x_n$ such that

$$X = G_{d,\epsilon}[x_1] \cup \cdots \cup G_{d,\epsilon}[x_n] .$$

Thus $(X,\underline{U})$ is totally bounded.

Next suppose that $(X,\underline{U})$ is totally bounded and complete. But then (by Definition 1) X, being a closed and totally bounded subset of itself, is compact. This completes the proof of Theorem 3.

Definition 2. A sequence x_n in a uniform space $(X,\underline{U})$ is called a <u>Cauchy sequence</u> if for each U in $\underline{U}$ there exists an integer N(U) such that $(x_n,x_m) \in U$ for all m and n greater than N(U).

Theorem 4. A Cauchy sequence in a uniform space converges to each of its cluster points.

Proof. From Theorem 1 of Section IIC it follows that for each point x in X the family

$$\{U[x] : U \in \underline{U}\}$$

is a base for the neighborhood system of x. Let x_n be a Cauchy sequence. Let U be a given member of $\underline{U}$. Then there exists a V in $\underline{U}$ such that $V \circ V \subset U$. Since x_n is a Cauchy sequence there exists an integer N(V) such that $(x_n,x_m) \in V$ for all m and n greater than N(V). Let x' be a cluster point of x_n. Then $V[x']$ contains infinitely many entries of the sequence x_n and therefore at least one, say x_k, such that $k > N(V)$. Then

$$(x',x_k) \in V \text{ and } (x_k,x_n) \in V$$

for all $n > N(V)$. Hence

$$(x', x_n) \in V \circ V \subset U \text{ for each } n > N(V).$$

Therefore $x_n \in U[x']$ for each $n > N(V)$. That is, x_n converges to its cluster point x'. This completes the proof.

Theorem 5. In a complete uniform space every Cauchy sequence converges.

Proof. Let $(X,\underline{U})$ be a complete uniform space and let x_n be a Cauchy sequence in X. Then the set

$$A = \{ y : y = x_n \text{ for some integer } n \}$$

is totally bounded. For let U be a given member of $\underline{U}$. Then, because x_n is Cauchy, there exists an integer N(U) such that $(x_n, x_m) \in U$ for all n and m greater than N(U). But then

$$A \subset U[x_1] \cup \cdots \cup U[x_{N+1}],$$

and therefore A is totally bounded. From Theorem 2 it follows that $\overline{A}$ is totally bounded. But then, since $(X,\underline{U})$ is complete, it follows from Definition 1 that $\overline{A}$ is compact. Hence by Theorem 5 of Section IIB x_n has a cluster point x'. But since x_n is a Cauchy sequence it follows from Theorem 4 that x_n converges to x'. Thus every Cauchy sequence in a complete uniform space converges. This completes the proof.

* Theorem 6. In a complete uniform space the closure of the range of an almost periodic movement is a compact set.

Proof. It follows from Theorem 12 of Section IIIA and from Theorem 2 of the present section that the closure of the range of $f(t)$ is totally bounded. Hence, by the definition of completeness (Definition 1) it follows that the closure of the range of $f(t)$ is a compact set.

Let $x = G(u_1, u_2, \ldots)$, x in X, be a function depending on an infinite sequence of variables, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \ldots$. If $\underline{u} = (u_1, u_2, \ldots)$ and $\underline{u}' = (u_1', u_2', \ldots)$ then $G(\underline{u}) = G(u_1, u_2, \ldots)$, $\underline{u} \cdot t = (u_1 t, u_2 t, \ldots)$, and $\underline{u} + \underline{u}' = (u_1 + u_1', u_2 + u_2', \ldots)$. A neighborhood of the vector $\underline{u}$ is defined as the set of vectors $\underline{u}'$ satisfying the inequalities

$$|u_\mu' - u_\mu| < \delta \qquad\qquad \mu = 1, \ldots, m$$

where m is a positive integer and $\delta > 0$.

Definition 3. A function $x = G(\underline{u})$, $-\infty < u_\nu < \infty$, $\nu = 1, 2, \ldots$, x in X is called <u>limit periodic</u> with the limit period 2π in each variable if to each U in $\underline{U}$ there corresponds a positive δ and positive integers m and N such that $(G(\underline{u}'), G(\underline{u}'')) \in U$ whenever $|u_\mu'' - u_\mu'| \leqq \delta \pmod{2N\pi}$, $\mu = 1, \ldots, m$.

Theorem 7. A limit periodic function $G(\underline{u})$ with the limit period 2π in each variable is uniformly continuous.

Proof. It must be shown that to each U in $\underline{U}$ there corresponds a positive δ and a positive integer m such that $(G(\underline{u}'), G(\underline{u}'')) \in U$ whenever $|u_\mu'' - u_\mu'| < \delta$, $\mu = 1, \ldots, m$. But this follows immediately from Definition 3 since the conditions $|u_\mu'' - u_\mu'| < \delta$, $\mu = 1, \ldots, m$ imply

$|u_\mu'' - u_\mu'| < \delta \pmod{2N\pi}$, $\mu = 1,\ldots,m$ for any given positive integer N.

Definition 4. Let R denote a compact separable metric space consisting of points denoted v, v_1, $v_2,\ldots$ and with neighborhoods $N(v)$. A function $x = G(\underline{u};v) = G(u_1, u_2,\ldots;v)$, $-\infty < u_\nu < \infty$, $\nu = 1,2,\ldots$, v in R and x in X, is called a <u>uniformly</u> <u>continuous</u> <u>family</u> <u>of</u> <u>limit</u> <u>periodic</u> <u>functions</u> if

(a) the function $G(\underline{u};v)$ is limit periodic with the limit period 2π in each variable u_μ for each v in R, and if

(b) to each U in $\underline{U}$ and v_o in R there corresponds a neighborhood $N_U(v_o)$ such that $(G(\underline{u};v_o), G(\underline{u};v)) \in U$ when $-\infty < u_\nu < \infty$, $\nu = 1,2,\ldots$, and $v \in N_U(v_o)$.

Theorem 8. If $G(\underline{u};v)$ is a uniformly continuous family of limit periodic functions then to each U in $\underline{U}$ corresponds a positive δ and positive integers m and N such that $(G(\underline{u}';v), G(\underline{u}'';v)) \in U$ whenever $|u_\mu'' - u_\mu'| \leqq \delta$ $\pmod{2N\pi}$, $\mu = 1,\ldots,m$; $v \in R$.

Proof. By Definition 3 and condition (a) of Definition 4 we can choose $\delta(v_o)$ and positive integers $m(v_o)$ and $N(v_o)$ such that

$$(G(\underline{u}';v_o), G(\underline{u}'';v_o)) \in F_{d,\epsilon/3}$$

when $|u_\mu'' - u_\mu'| \leqq \delta(v_o) \pmod{2\pi N(v_o)}$, $\mu = 1,\ldots, m(v_o)$. By condition (b) of Definition 4 it follows that

$$
\begin{aligned}
d(G(\underline{u}';v),\ G(\underline{u}'';v)) &\leqq d(G(\underline{u}';v),\ G(\underline{u}';v_o)) \\
&\quad + d(G(\underline{u}';v_o),\ G(\underline{u}'';v_o)) \\
&\quad + d(G(\underline{u}'';v_o),\ G(\underline{u}'';v)) \\
&\leqq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \ ,
\end{aligned}
$$

whenever $|u_\mu'' - u_\mu'| \leqq \delta(v_o) \pmod{2\,N(v_o)\,\pi}$, $\mu = 1,\ldots,m(v_o)$, and $v \in N_{F_d,\,\epsilon/3}(v_o)$. Since R is compact we can choose $v_1,\ldots,v_n \in R$ such that

$$R = N_{F_d,\,\epsilon/3}(v_1) \cup \cdots \cup N_{F_d,\,\epsilon/3}(v_n) \ .$$

It follows that the statement of Theorem 8 holds if we choose

$$\delta = \min \{ \delta(v_\nu) : \nu = 1,\ldots,n \}$$

$$m = \max \{ m(v_\nu) : \nu = 1,\ldots,n \}$$

$$N = N(v_1) \cdot\cdot\cdot N(v_n) \ .$$

This completes the proof.

Theorem 9. Let $G(\underline{u};v)$ be a uniformly continuous family of limit periodic functions and let $\underline{\lambda} = (\lambda_1, \lambda_2,\ldots)$ be a constant real vector. Then the function $f(t;v) = G(\underline{\lambda}t;v)$ is a uniformly continuous family of almost periodic movements.

Proof. From Theorem 8 it follows that a real number τ is a U-translation number of every function $f(t;v_o)$, v_o in R, if the inequalities $|\lambda_\mu \tau| \leqq \delta \pmod{2\,N\,\pi}$, $\mu = 1,\ldots,m$ are satisfied. But the set of numbers τ satisfying this inequality is a $(\delta/N; \lambda/N)$-neighborhood of zero, and by Theorem 8 of Section IIA this set is relatively dense. Hence condition (a) of Definition 6 of Section IIIA is satisfied. Condition (b) of Definition 6 of Section IIIA is an immediate consequence of condition (b) of Definition 4 of this section.

*Theorem 10. Let $x_1 = G_1(\underline{u})$, $x_2 = G_2(\underline{u})$, x_1 and x_2 in X, be two limit periodic functions with the limit period 2π and let $\underline{\beta} = (\beta_1, \beta_2, \ldots)$ be a real vector with linearly independent coordinates. If $d(G_1(\underline{\beta}t), G_2(\underline{\beta}t)) = 0$ for all real t, then $d(G_1(\underline{u}), G_2(\underline{u})) = 0$ for all $\underline{u}$.

Proof. According to Definition 3 we can choose numbers δ_1, m_1, and N_1 corresponding to $G_1(\underline{u})$, $\epsilon/2$, $d \in \underline{G}$ and also numbers δ_2, m_2, and N_2 corresponding to $G_2(\underline{u})$, $\epsilon/2$, and $d \in \underline{G}$ such that

$$d(G_1(\underline{u}'), G_1(\underline{u}'')) \leqq \epsilon/2 \text{ whenever } |u_\mu'' - u_\mu'| \leqq \delta_1 \pmod{2 N_1 \pi} \quad \mu = 1, \ldots, m_1$$

and

$$d(G_2(\underline{u}'), G_2(\underline{u}'')) \leqq \epsilon/2 \text{ whenever } |u_\mu'' - u_\mu'| \leqq \delta_2 \pmod{2 N_2 \pi} \quad \mu = 1, \ldots, m_2 .$$

Let $\delta = \min(\delta_1, \delta_2)$, $m = \max(m_1, m_2)$, $N = N_1 \cdot N_2$. Then

$$\begin{aligned} d(G_1(\underline{u}), G_2(\underline{u})) \leqq\ & d(G_1(\underline{u}), G_1(\underline{\beta}t)) \\ & + d(G_1(\underline{\beta}t), G_2(\underline{\beta}t)) \\ & + d(G_2(\underline{\beta}t), G_2(\underline{u})) \\ \leqq\ & \epsilon/2 + 0 + \epsilon/2 = \epsilon , \end{aligned}$$

whenever $|\beta_\mu t - u_\mu| \leqq \delta \pmod{2N\pi}$, $\mu = 1,\ldots,m$. But it follows from Kronecker's theorem (Theorem 6 of Section IIA) that such a number t can be chosen for each selection of values of $u_1,\ldots,u_m$. Hence

$$d(G_1(\underline{u}), G_2(\underline{u})) \leqq \epsilon$$

for all $\underline{u}$ and for arbitrary positive ϵ and arbitrary d in $\underline{G}$. Therefore, for each d in $\underline{G}$, we have

$$d(G_1(\underline{u}), G_2(\underline{u})) = 0$$

for all $\underline{u}$. This completes the proof.

* Theorem 11. Let $x_1 = G_1(\underline{u})$, $x_2 = G_2(\underline{u})$, x_1 and x_2 in X, be two limit periodic functions with the limit period 2π and let $\underline{\beta} = (\beta_1, \beta_2,\ldots)$ be a real vector with linearly independent coordinates. If $(X,\underline{U})$ is Hausdorff and if the almost periodic movements $G_1(\underline{\beta}t)$ and $G_2(\underline{\beta}t)$ are identical, then the functions $G_1(\underline{u})$ and $G_2(\underline{u})$ are also identical.

Proof. This theorem follows immediately from the preceding theorem and from Theorem 3 (d) of Section IIC.

Definition 5. Let $x = G(\underline{u};v)$ be a uniformly continuous family of limit periodic functions and let $\underline{\beta} = (\beta_1, \beta_2,\ldots)$ be a real vector with linearly independent coordinates. The uniformly continuous family $x = f(t;v) = G(\underline{\beta}t;v)$ of almost periodic movements is called the <u>diagonal family</u> of $G(\underline{u};v)$ corresponding to $\underline{\beta}$ and the family $x = G(\underline{u};v)$ is called a <u>spatial extension</u> of $f(t;v)$ corresponding to $\underline{\beta}$.

If $(X,\underline{U})$ is a Hausdorff uniform space then it follows from Theorem 11 that the spatial extension of a uniformly continuous family of almost periodic movements $x = f(t;v)$ corresponding to a vector $\underline{\beta}$ is uniquely determined by $f(t;v)$ and $\underline{\beta}$. On the other hand, even if $(X,\underline{U})$ is Hausdorff, the spatial extension does not always exist, but we shall prove that we can choose the vector $\underline{\beta}$ such that a spatial extension of $f(t;v)$ corresponding to $\underline{\beta}$ exists. For the proof we use the notion of a rational basis of a uniformly continuous family of almost periodic movements.

Definition 6. A sequence $\beta_1, \beta_2, \ldots$ of linearly independent real numbers is called a <u>basis</u> of $x = f(t;v)$ if to each U in $\underline{U}$ there corresponds a positive number η and positive integers M and N such that every real number τ satisfying the conditions

$$|\beta_\mu \tau| \leqq \eta \pmod{2 N \pi} \quad \mu = 1,\ldots,M \tag{1}$$

is a common U-translation number of all almost periodic functions of the family $x = f(t;v)$.

Theorem 12. Every uniformly continuous family of almost periodic movements in $(X,\underline{U})$ has a basis.

Proof. Let $f(t;v)$ be a uniformly continuous family of almost periodic movements, and let U be a given member of $\underline{U}$. Then there exists a $F_{d,\epsilon} \subset U$. According to Theorem 11 of Section IIIA there corresponds to each positive ϵ and pseudo-metric d in the gage G a $(\delta;\lambda)$-neighborhood of zero such that every number in this neighborhood is a common $(\epsilon;d)$-translation number of all the fuctions $f(t;v)$, for v in R. That is, to each positive ϵ and pseudo-metric d in G there corresponds a positive δ and real numbers $\lambda_1,\ldots,\lambda_m$ such that every number τ satisfying

$$|\lambda_\mu \tau| \leqq \delta \pmod{2\pi} \qquad \mu = 1,\ldots,m$$

is a common $(\epsilon;d)$-translation number of all the functions $f(t;v)$. Thus, in particular, for each $\epsilon_n = 1/n$, $n = 1,2,\ldots$, there corresponds a positive δ_n and real numbers $\lambda_1^{(n)},\ldots,\lambda_{m_n}^{(n)}$ such that every number τ satisfying

$$|\lambda_\mu^{(n)} \tau| \leqq \delta_n \pmod{2\pi} \qquad \mu = 1,\ldots,m_n$$

is a common $(1/n;d)$-translation number of all the functions $f(t;v)$ for v in R. If we arrange all the numbers $\lambda_\mu^{(n)}$ in a single sequence

$$\lambda_1^{(1)}, \lambda_2^{(1)},\ldots, \lambda_{m_1}^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)},\ldots, \lambda_{m_2}^{(2)}, \lambda_1^{(3)},\ldots$$

and denote them by

$$\lambda_1, \lambda_2, \lambda_3, \ldots$$

we can state Theorem 11 of Section IIIA in the form: To each positive ϵ and pseudo-metric d in G there corresponds a positive δ and an integer m such that every number τ satisfying

$$|\lambda_\mu \tau| \leqq \delta \pmod{2\pi} \qquad \mu = 1,\dots,m \qquad (2)$$

is a common $(\epsilon;d)$-translation number of all the functions $f(t;v)$ for v in R.

According to Theorem 5 of Section IIA we can find a sequence $\beta_1, \beta_2, \dots$ of linearly independent numbers such that every λ_μ has a representation

$$\lambda_\mu = r_{\mu 1}\beta_1 + \dots + r_{\mu q_\mu} \beta_{q_\mu} ,$$

where the numbers $r_{\mu\nu}$ are rational. Let $M = \max \{ q_\mu : \mu = 1,\dots,m \}$, let N denote the common denominator of the numbers $r_{\mu\nu}$, $1 \leqq \mu \leqq m$, $1 \leqq \nu \leqq q_\mu$, and let η be a real number chosen such that

$$0 < \eta < \min_{1 \leqq \mu \leqq m} \left\{ \frac{\delta}{|r_{\mu 1}| + \dots + |r_{\mu q_\mu}|} \right\} .$$

It will now be shown that with this choice of the constants M, N and η the inequalities (2) follow from the inequalities (1), hence proving the theorem. Now the inequalities (1) may be written

$$|\beta_\nu \tau - k_\nu 2N\pi| \leqq \eta \qquad \nu = 1,\dots,M,$$

where the k_ν are suitable integers. But then for each $\mu \leqq m$ we have

$$|r_{\mu\nu} \beta_\nu \tau - r_{\mu\nu} k_\nu 2N\pi| \leqq \eta |r_{\mu\nu}|$$
$$\nu = 1,\dots,q_\mu \leqq M.$$

Thus

$$\left| \lambda_\mu \tau - p_\mu 2\pi \right| = \left| \sum_{\nu=1}^{q_\mu} r_{\mu\nu}\beta_\nu \tau - \left(\sum_{\nu=1}^{q_\mu} r_{\mu\nu} k_\nu N\right) 2\pi \right|$$

$$\leqq \sum_{\nu=1}^{q_\mu} \left| r_{\mu\nu}\beta_\nu \tau - r_{\mu\nu} k_\nu N 2\pi \right|$$

$$\leqq \eta \sum_{\nu=1}^{q_\mu} |r_{\mu\nu}|$$

$$\leqq \frac{\delta \sum_{\nu=1}^{q_\mu} |r_{\mu\nu}|}{\max\limits_{1 \leqq \mu \leqq m} \left\{ \sum_{\nu=1}^{q_\mu} | r_{\mu\nu}| \right\}}$$

$$\leqq \delta \ , \ 1 \leqq \mu \leqq m,$$

where the $p_\mu = \sum_{\nu=1}^{q_\mu} r_{\mu\nu} N k_\nu$ are integers because of the choice of N. Thus we have

$$| \lambda_\mu \tau | \leqq \delta \ (\mathrm{mod}\ 2\pi) \qquad \mu = 1,\ldots,m,$$

which are the inequalities (2). But

$$F_{d,\epsilon} \subset U \text{ implies } \{\tau_f(\epsilon;d)\} \subset \{\tau_f(U)\} \ .$$

This completes the proof of Theorem 12.

Theorem 13. Let $(X,\underline{U})$ be a complete uniform space. If the sequence $\underline{\beta} = (\beta_1, \beta_2, \ldots)$ is a basis of a uniformly continuous family of almost periodic movements $f(t;v)$, then there exists a spatial extension of $f(t;v)$ corresponding to $\underline{\beta} = (\beta_1, \beta_2, \ldots)$.

Proof. Let $\underline{u} = (u_1, u_2, \ldots)$ be any given sequence of real numbers. If n is a positive integer it follows from Kronecker's theorem that we can find a real number $t_n(\underline{u})$ such that

$$|\beta_\nu t_n(\underline{u}) - u_\nu| \leqq 1/n \pmod{n!2\pi}, \qquad \nu = 1, \ldots, n .$$

If p is a positive integer it follows that

$$|\beta_\nu (t_{n+p}(\underline{u}) - t_n(\underline{u}))| \leqq 2/n \pmod{n!2\pi}, \qquad \nu = 1, \ldots n . \tag{3}$$

By comparing (3) with the conditions (1) of Definition 6 we find that $t_{n+p}(\underline{u}) - t_n(\underline{u})$ is a common U-translation number of all the functions $f(t;v)$ for v in R if and only if

$$n \geqq \max(2/\eta, N, M) .$$

Hence we have

$$\begin{aligned}(f(t_{n+p}(\underline{u});v), &f(t_n(\underline{u});v)) \\ &= (f(t_n(\underline{u}) + \{t_{n+p}(\underline{u}) - t_n(\underline{u})\};v), \\ &\quad f(t_n(\underline{u});v)) \in U\end{aligned}$$

for all positive integers p, provided only that

$$n \geqq \max(2/\eta, N, M) .$$

That is $f(t_n(\underline{u});v)$ $n = 1,2,\ldots$ is a Cauchy sequence in $(X,\underline{U})$. Since $(X,\underline{U})$ was assumed to be complete, it follows from Theorem 5 that $f(t_n(\underline{u});v)$ converges. That is, for each

$\underline{u} = (u_1, u_2, \ldots)$ and v in R there exists a point $G(\underline{u};v)$ such that for every $\epsilon > 0$ and d in the gage $\underline{G}$ there corresponds an integer $m = m(\underline{u}, v, \epsilon, d)$ such that

$$d(G(\underline{u};v), f(t_n(\underline{u});v)) \leqq \epsilon$$

for all $n > m$. Note that if $(X,\underline{U})$ is Hausdorff, then $G(\underline{u};v)$ is uniquely determined by $\underline{u}$ and v. In any case, Hausdorff or not,

$$d(G(\underline{u};v), G'(\underline{u};v)) = 0$$

for all d in $\underline{G}$ if $f(t_n(\underline{u});v)$ converges to both $G(\underline{u};v)$ and $G'(\underline{u};v)$.

We can define a single-valued function $G(\underline{u};v)$ by selecting for its value at $(\underline{u};v)$ any one of the points of X to which $f(t_n(\underline{u});v)$ converges. We now prove that each such single-valued function $G(\underline{u};v)$ is a uniformly continuous family of limit periodic functions. Let $\epsilon > o$ and $d \in \underline{G}$ be given. According to Definition 6 we can find a corresponding $\eta > o$ and positive integers M and N such that every real τ satisfying

$$|\beta_\mu \tau| \leqq \eta (\text{mod } 2 N \pi(, \qquad \mu = 1,\ldots,M$$

is a common $(\epsilon;d)$-translation number of all the functions $f(t;v)$ for v in R. If $\underline{u}'$ and $\underline{u}''$ are two vectors satisfying

$$|u''_\nu - u'_\nu| \leqq \eta/3 \ (\text{mod } 2N\pi) \qquad \nu=1,\ldots,M \tag{4}$$

we have for $n \geqq \max(M,N)$ that the following inequalities hold mod $2N\pi$ when $\nu = 1,\ldots,M$.

$$
\begin{aligned}
&\left| \beta_\nu (t_n(\underline{u}'') - t_n(\underline{u}')) \right| \\
&\leqq \left| \beta_\nu t_n(\underline{u}'') - u_\nu'' \right| + \left| u_\nu'' - u_\nu' \right| + \left| u_\nu' - \beta_\nu t_n(\underline{u}') \right| \\
&\leqq \eta/3 + 2/n \,.
\end{aligned}
$$

If we further choose $n \geqq 3/\eta$ we obtain

$$\left| \beta_\nu (t_n(\underline{u}'') - t_n(\underline{u}')) \right| \leqq \eta \pmod{2N\pi}$$

for $\nu = 1,\ldots,M$. Thus the inequalities (1),

$$\left| \beta_\mu \tau \right| \leqq \eta \pmod{2N\pi}\,, \qquad \mu = 1,\ldots,M, \tag{1}$$

of Definition 6 are satisfied by the numbers $t_n(\underline{u}'') - t_n(\underline{u}')$ for n sufficiently large, and we therefore have

$$
\begin{aligned}
&d(f(t_n(\underline{u}'');v),\ f(t_n(\underline{u}');v)) \\
&= d(f(t_n(\underline{u}') + \{t_n(\underline{u}'') - t_n(\underline{u}')\}\,;v),\ f(t_n(\underline{u}');v)) \\
&\leqq \epsilon \,.
\end{aligned}
$$

It follows that for sufficiently large n

$$
\begin{aligned}
&d(G(\underline{u}'';v),\ G(\underline{u}';v)) \\
&\leqq d(G(\underline{u}'';v),\ f(t_n(\underline{u}'');v)) + d(f(t_n(\underline{u}'');v),\ f(t_n(\underline{u}');v)) \\
&+ d(f(t_n(\underline{u}');v),\ G(\underline{u}';v)) \leqq 3\epsilon
\end{aligned}
$$

whenever (4) is satisfied. This proves that condition (a) of Definition 4 is satisfied.

If $v_o \varepsilon R$ is given, it follows from condition (b) of Definition 6 of Section IIIA that there exists a neighborhood $N_{F_{d,\epsilon}}(v_o)$ such that

$$d(f(t;v_o),\ f(t;v)) \leqq \epsilon$$

for all real t and all v in $N_{F_{d,\epsilon}}(v_o)$. Thus in particular

$$d(f(t_n(\underline{u});v_o),\ f(t_n(\underline{u});v)) \leqq \epsilon$$

for all $\underline{u}$ and n whenever v is in $N_{F_{d,\epsilon}}(v_o)$. Hence

$$\begin{aligned} d(G(\underline{u};v_o),\ G(\underline{u};v)) &\leqq d(G(\underline{u};v_o),\ f(t_n(\underline{u});v_o)) \\ &\quad + d(f(t_n(\underline{u});v_o),\ f(t_n(\underline{u});v)) \\ &\quad + d(f(t_n(\underline{u});v),\ G(\underline{u};v)) \leqq 3\epsilon \end{aligned}$$

for all v in $N_{F_{d,\epsilon}}(v_o)$ if n is chosen sufficiently large. This proves that condition (b) of Definition 4 is satisfied and the function $G(\underline{u};v)$ is a uniformly continuous family of limit periodic functions.

By Theorem 9 $G(\underline{\beta}t;v)$ is a uniformly continuous family of almost periodic movements. Thus $G(\underline{u};v)$ will be a spatial extension of $f(t;v)$ corresponding to $\underline{\beta}$ if $f(t;v) = G(\underline{\beta}t;v)$ for all real t and all v in R. For $\underline{u} = \underline{\beta}t$,

$$\left| \beta_\nu t_n(\underline{u}) - u_\nu \right| \leqq 1/n \pmod{n!2\pi} \quad \nu = 1,\ldots,n$$

will be satisfied if we choose $t_n(\underline{\beta}t) = t$ for each n. But then the sequence

$$f(t_n(\underline{\beta}t);v) = f(t;v) \qquad n = 1,2,\ldots$$

converges to $f(t;v)$. Hence, for $\underline{u} = \underline{\beta}t$, the value of $G(\underline{u};v)$ can be chosen to be $f(t;v)$. That is, there exists a single-valued function $G(\underline{u};v)$ as defined before which is equal to $f(t;v)$ for $\underline{u} = \underline{\beta}t$. This completes the proof of Theorem 13.

Theorem 14. To a uniformly continuous family $G(\underline{u};v)$ of limit periodic functions and a given member U of the uniformity $\underline{U}$ there corresponds a uniformly continuous family $G^*(\underline{u};v) = G^*(u_1,\ldots,u_m;v)$ of limit periodic functions depending on only a finite number of variables and satisfying

$$(G^*(\underline{u};v)\ ,\ G(\underline{u};v)) \varepsilon U$$

for all $\underline{u}$ and v.

Proof. According to Theorem 8 there corresponds to the given U a positive number δ and positive integers m and N such that

$$(G(\underline{u}';v),\ G(\underline{u}'';v))\ \varepsilon\ U$$

whenever $|u''_\mu - u'_\mu| \leqq \delta (\text{mod } 2\ N\ \pi)$, $\mu = 1,\ldots,m$, and v is in R. Define $G^*(\underline{u};v)$ to be $G(u_1,\ldots,u_m,o,o,\ldots;v)$. Then

$$(G^*(\underline{u};v),\ G(\underline{u};v))\ \varepsilon\ U$$

for all $\underline{u}$ and v.

* Theorem 15. If $(X,\underline{U})$ is a complete uniform space, then the closure of the range of a uniformly continuous family

of limit periodic functions is a compact set.

Proof. By Definition 1 and Theorem 2 it suffices to show that the range is totally bounded. Let $G(\underline{u};v)$ be the uniformly continuous family of limit periodic functions and let U be an arbitrarily given member of the uniformity $\underline{U}$. Then there exists a positive ϵ and a pseudo-metric d such that $F_{d,\epsilon} \subset U$. According to Theorem 14 there corresponds to $G(\underline{u};v)$ and $F_{d,\epsilon/3}$ a uniformly continuous family $G^*(\underline{u};v)$ of limit periodic functions depending on only a finite number of variables and satisfying

$$d(G^*(\underline{u};v),\ G(\underline{u};v)) \leqq \epsilon/3 \tag{5}$$

for all $\underline{u}$ and v. Since the domain of $G^*(\underline{u};v)$ is the compact set

$$0 \leqq u_\mu \leqq 2\,N\,\pi, \qquad \mu = 1,\ldots,m,\ v \in R$$

it follows from Theorem 3 of Section IIB that the range of $G^*(\underline{u};v)$ is compact, and hence by Theorem 3 of this section the range of $G^*(\underline{u};v)$ is totally bounded. Consequently there exist a finite number of points

$$(\underline{u}^{(1)};v^{(1)}),\ldots,(\underline{u}^{(n)};v^{(n)}),$$

$0 \leqq u_\nu^{(j)} \leqq 2\,N\,\pi$, such that for each $\underline{u}$ and v

$$d(G^*(\underline{u};v),\ G^*(\underline{u}^{(k)};v^{(k)})) \leqq \epsilon/3 \tag{6}$$

for at least one k, $1 \leqq k \leqq n$. Thus it follows from (5) and (6) that for each $\underline{u}$ and v

$$d(G(\underline{u};v),\ G(\underline{u}^{(k)};v^{(k)})) \leqq \epsilon$$

for at least one k, $1 \leqq k \leqq n$. That is,

$$\begin{aligned} G(\underline{u};v) &\in F_{d,\epsilon}[G(\underline{u}^{(1)};v^{(1)})] \cup \cdots \cup F_{d,\epsilon}[G(\underline{u}^{(n)};v^{(n)})] \\ &= F_{d,\epsilon}[\{G(\underline{u}^{(1)};v^{(1)}),\ldots,G(\underline{u}^{(n)};v^{(n)})\}] \\ &\subset U[\{G(\underline{u}^{(1)};v^{(1)}),\ldots,G(u^{(n)};v^{(n)})\}] \end{aligned}$$

for all $\underline{u}$ and v. Thus the range of $G(\underline{u};v)$ is totally bounded. This completes the proof.

C. The Approximation Theorem

Definition 1. A uniform space $(X,\underline{U})$ is called <u>continuously locally arcwise connected</u> if to each compact subset C of X there corresponds a U_C in $\underline{U}$ and a continuous function $z = \emptyset(x;t;y)$ defined on $0 \leqq t \leqq 1$, and (x,y) in $U_C \cap C \times C$, with z in X such that

$$\emptyset(x;o;y) = x,\ \emptyset(x;1;y) = y, \text{ and } \emptyset(x;t;x) = x.$$

* Theorem 1. Let $(X,\underline{U})$ be a continuously locally arcwise connected uniform space and let C be a compact subset of X. Then to each U in $\underline{U}$ there corresponds a V in $\underline{U}$ such that $(x, \emptyset(x;t;y)) \in U$ whenever $o \leqq t \leqq 1$ and (x,y) is in $V \cap C \times C$.

Proof. Let U be a given member of the uniformity $\underline{U}$. By Theorem 3b of Section IIC there exists a closed member of $\underline{U}$, say F, such that $F \subset U_C$. By Theorem 1 of Section IIB, $C \times C$ is compact. Since $F \cap C \times C$ is a closed subset of $C \times C$, it follows from Theorem 2 of Section IIB that $F \cap C \times C$ is compact. It then follows from Theorem 1 of Section IIB and Theorem 2 of Section IIC that $\emptyset(x;t;y)$ is uniformly continuous for $o \leqq t \leqq 1$ and (x,y) in $F \cap C \times C \subset U_C \cap C \times C$. Thus there exists a V' in $\underline{U}$ corresponding to U in $\underline{U}$ such that

$$(\emptyset(x;t;x), \emptyset(x;t;y)) \in U$$

whenever (x,y) is in $V' \cap F \cap C \times C$ and $o \leqq t \leqq 1$. But $V' \cap F$ is a member of $\underline{U}$, and so, since $\emptyset(x;t;x) = x$, the theorem is proved if we choose $V = V' \cap F$.

Throughout this section it will be assumed that $(X,\underline{U})$ is a complete and continuously locally arcwise connected uniform space. In general it will not be assumed that $(X,\underline{U})$ is Hausdorff.

Definition 2. Let $G(\underline{u};v)$ be a uniformly continuous family of limit periodic functions with values in X. If we give the variable u_ν a fixed value α we obtain a function which we shall denote

$$G_{\nu;\alpha}(\underline{u};v) = G(u_1,\ldots,u_{\nu-1},\alpha,u_{\nu+1},\ldots;v) .$$

Definition 3. Let C be the compact closure of the range of $G(\underline{u};v)$. Let $U_C \in \underline{U}$ and $\emptyset(x;t;y)$ be as in Definition 1.

According to Theorem 8 of Section IIIB there corresponds to U_C positive δ and a positive integer N such that

$$(G(\underline{u}';v),\ G(\underline{u}'';v))\ \varepsilon\ U_C$$

whenever $|u''_\nu - u'_\nu| \leqq \delta \pmod{2N\pi}$ and $u'_\mu = u''_\mu$ for $\mu \neq \nu$. We define

$$H(\underline{u};v) = T_{\nu;N,\delta}\ G(\underline{u};v)$$

in the following way:

$$H(\underline{u};v) = G(\underline{u}:v) \quad \text{when } o \leqq u_\nu \leqq 2N\pi - \delta .$$

$$H(\underline{u};v) = \emptyset(G_{\nu;2N\pi-\delta}(\underline{u};v);\frac{u_\nu - 2N\pi + \delta}{\delta};G_{\nu;o}(\underline{u};v))$$
$$\text{when } 2N\pi - \delta \leqq u_\nu \leqq 2N\pi .$$

$H(\underline{u};v)$ is periodic in u_ν with the period $2N\pi$.

The function $H(\underline{u};v) = T_{\nu;N,\delta}\,G(\underline{u};v)$ is called a <u>periodification of</u> $G(\underline{u};v)$ <u>with respect to the variable</u> u_ν.

* Theorem 2. The periodification $H(\underline{u};v) = T_{\nu;N,\delta}\,G(\underline{u};v)$ is a uniformly continuous family of limit periodic functions.

Proof. The function $\emptyset(x;t;y)$ is uniformly continuous for $o \leqq t \leqq 1$ and (x,y) in $F \cap C \times C$ where F is a closed member of $\underline{U}$ such that $F \subset U_C$. (Cf. the proof of Theorem 1.) Thus for each given positive ϵ and pseudo-metric d in the gage $\underline{G}$ there exist two positive numbers η_1 and η_2, and a pseudo-metric p in $\underline{G}$ such that

$$d(\emptyset(x;t_1;y),\ \emptyset(x;t_2;y)) \leqq \epsilon/4 \quad \text{for } |t_2 - t_1| \leqq n_1 \ , \qquad (1)$$

and

$$d(\emptyset(x_1;t;y_1),\ \emptyset(x_2;t;y_2)) \leqq \epsilon/2 \qquad (2)$$

for $p(x_1,x_2) \leqq n_2$ and $p(y_1,y_2) \leqq n_2$, where each pair (x,y) belongs to $F \cap C \times C$, and $o \leqq t \leqq 1$ for each t. According to Theorem 8 of Section IIIB we can choose a positive number $\delta_1 \leqq n_1 \delta$ such that

$$d(G(\underline{u}';v),\ G(\underline{u}'';v)) \leqq \epsilon/4 \qquad (3)$$

when $|u''_\nu - u'_\nu| \leqq \delta_1$, $u''_\mu = u'_\mu$ for $\mu \neq \nu$, and v is in R. It can now be shown that

$$d(H(\underline{u}';v),\ H(\underline{u}'';v)) \leqq \epsilon/2 \qquad (4)$$

when $|u''_\nu - u'_\nu| \leqq \delta_1 \pmod{2 N \pi}$, $u''_\mu = u'_\mu$ for $\mu \neq \nu$, and v is in R: Since $H(\underline{u};v)$ is periodic with period $2 N \pi$ we need consider only the following three cases in order to establish (4):

(a) u''_ν and u'_ν both lie in the interval $[o,\ 2 N \pi - \delta]$, $|u''_\nu - u'_\nu| \leqq \delta_1$, and $u'_\mu = u''_\mu$ for $\mu \neq \nu$. Then $d(H(\underline{u}';v),\ H(\underline{u}'';v)) = d(G(\underline{u}';v),\ G(\underline{u}'';v)) \leqq \epsilon/4$ by (3).

(b) u''_ν and u'_ν both lie in the interval $[2 N \pi - \delta,\ 2 N \pi]$, $|u'_\nu - u''_\nu| \leqq \delta_1$, and $u'_\mu = u''_\mu$ for $\mu \neq \nu$. In this case

$$d(H(\underline{u}'';v),\ H(\underline{u}';v)) =$$

$$d(\emptyset(G_{\nu;2N\pi-\delta}(\underline{u}'';v);\frac{u_\nu'' - 2N\pi+\delta}{\delta};G_{\nu;o}(\underline{u}'';v)),$$

$$\emptyset(G_{\nu;2N\pi-\delta}(\underline{u}';v);\frac{u_\nu' - 2N\pi+\delta}{\delta};G_{\nu;o}(\underline{u}';v)))$$

$$\leqq \epsilon/4,$$

since $\left|\frac{u_\nu'' - 2N\pi+\delta}{\delta} - \frac{u_\nu' - 2N\pi+\delta}{\delta}\right| \leqq \frac{\delta_1}{\delta} \leqq n_1$, and (1).

(c) u_ν'' lies in $[o,\ 2N\pi-\delta]$ and u_ν' lies in $[2N\pi-\delta,\ 2N\pi]$, $|u_\nu'' - u_\nu'| \leqq \delta_1$ and $u_\mu' = u_\mu''$ for $\mu \neq \nu$. In this case

$$d(H(\underline{u}'';v),\ H(\underline{u}';v)) =$$

$$d(G(\underline{u}'';v),\ \emptyset(G_{\nu;2N\pi-\delta}(\underline{u}';v);\frac{u_\nu' - 2N\pi+\delta}{\delta};G_{\nu;o}(\underline{u}';v)))$$

$$\leqq d(G(\underline{u}''v),G_{\nu;2N\pi-\delta}(\underline{u}';v))$$

$$+\ d(G_{\nu;2N\pi-\delta}(\underline{u}';v),\ \emptyset(G_{\nu;2N\pi-\delta}(\underline{u}';v);\frac{u_\nu' - 2N\pi+\delta}{\delta};$$

$$G_{\nu;o}(\underline{u}';v))) \leqq \epsilon/4 \text{ (by (3))}$$

$$+\ d(\emptyset(G_{\nu;2N\pi-\delta}(\underline{u}';v);0;G_{\nu;o}(\underline{u}';v)),$$

$$\emptyset(G_{\nu;2N\pi-\delta}(\underline{u}';v);\frac{u_\nu' - 2N\pi+\delta}{\delta};G_{\nu;o}(\underline{u}';v)))$$

$\leqq \epsilon/4 + \epsilon/4 = \epsilon/2$ by (1) and (3), since in this case

$$|u_\nu' - 2N\pi+\delta| \leqq |u_\nu' - u_\nu''| \leqq \delta_1 \leqq n_1\delta.$$

Thus (4) is established.

Next we choose (again by Theorem 8 of Section IIIB) a positive number δ_2 and positive integers N^* and $m \geqq \nu$ corresponding to $F_{d,\epsilon/2} \cap F_{p,\eta_2} \varepsilon \underline{U}$ such that

$$\begin{aligned} &d(G(\underline{u}';v),\ G(\underline{u}'';v)) \leqq \epsilon/2, \text{ and} \\ &p(G(\underline{u}';v),\ G(\underline{u}'';v)) \leqq \eta_2 \end{aligned} \tag{5}$$

when $|u''_\mu - u'_\mu| \leqq \delta_2 \pmod{2N^*\pi}$ $\mu=1,\ldots,m$, and v is in R. It follows immediately because of (2) and (5) and the definition of $H(\underline{u};v)$ that in any case

$$d(H(\underline{u}';v),\ H(\underline{u}'';v)) \leqq \epsilon/2$$

when $u'_\nu = u''_\nu$, $|u''_\mu - u'_\mu| \leqq \delta_2 \pmod{2N^*\pi}$, $\mu = 1,\ldots,m$, v in R. Letting $\underline{u}'_{\nu''}$ denote the vector $(u'_1,\ldots,u'_{\nu-1}, u''_\nu, u'_{\nu+1},\ldots)$ and combining (4) and (5) we obtain

$$\begin{aligned} d(H(\underline{u}';v),\ H(\underline{u}'';v)) &\leqq d(H(\underline{u}';v),\ H(\underline{u}'_{\nu''};v)) \\ &\quad + d(H(\underline{u}'_{\nu''};v),\ H(\underline{u}'';v)) \\ &\leqq \epsilon/2 + \epsilon/2 = \epsilon\ , \end{aligned}$$

when $|u''_\mu - u'_\mu| \leqq \min(\delta_1, \delta_2) \pmod{2NN^*\pi}$, $\mu = 1,\ldots,m$, and v in R. Thus $H(\underline{u};v)$ satisfies condition (a) of Definition 4 of Section IIIB. That is, $H(\underline{u};v)$ is limit periodic for each v in R.

Let v_o be a given point of R. Then we can choose a neighborhood $N(v_o)$ corresponding to $F_{d,\epsilon} \cap F_{p,\eta_2} \varepsilon \underline{U}$ such that

$$d(G(\underline{u};v_o),\ G(\underline{u};v)) \leqq \epsilon \quad \text{and}$$
$$p(G(\underline{u};v_o),\ G(\underline{u};v)) \leqq \eta_2 \tag{6}$$

when v is in $N(v_o)$ and for all vectors $\underline{u}$. It follows from (2) and (6), and the definition of $H(\underline{u};v)$ that

$$d(H(\underline{u};v_o),\ H(\underline{u};v)) \leqq \epsilon$$

whenever v is in $N(v_o)$. That is, $H(\underline{u},v)$ satisfies condition (b) of Definition 4 of Section IIIB, and $H(\underline{u};v)$ is consequently a uniformly continuous family of limit periodic functions.

Theorem 3. If the family $G(\underline{u};v)$ is periodic with the period $2\ N_1\ \pi$ in the variable u_μ ($\mu \neq \nu$), then the function $H(\underline{u};v) = T_{\nu;N,\delta}\ G(\underline{u};v)$ also has the period $2\ N_1\ \pi$ in the variable u_μ. If $G(\underline{u};v)$ is independent of u_μ ($\mu \neq \nu$), the function $H(\underline{u};v)$ is also independent of u_μ.

Proof. The theorem follows immediately from the definition of $H(\underline{u};v)$.

* Theorem 4. To each U in $\underline{U}$ there corresponds a positive δ and a positive integer N such that the family $H(\underline{u};v) = T_{\nu;N,\delta}\ G(\underline{u};v)$ satisfies the condition

$$(H(\underline{u};v),\ G(\underline{u};v)) \ \varepsilon\ U \tag{7}$$

for every vector $\underline{u}$ and every v in R.

Proof. According to Theorem 15 of Section IIIB the closure of the range of $G(\underline{u};v)$ is a compact set C. Let U be a given member of the uniformity $\underline{U}$. Then there exists a $F_{d,\epsilon} \subset U$. In turn there corresponds to $F_{d,\epsilon}$ according to Theorem 1 a $F_{p,\eta}$ such that

$$d(x, \phi(x;t;y)) \leqq \epsilon/2 \qquad (8)$$

whenever $0 \leqq t \leqq 1$ and (x,y) is in $F_{p,\eta} \cap C \times C$. Furthermore, according to Theorem 8 of Section IIIB we can choose a positive δ and a positive integer N such that

$$(G(\underline{u}';v), G(\underline{u}'';v)) \ \varepsilon\ F_{d,\epsilon/2} \cap F_{p,\eta} \qquad (9)$$

whenever $u'_\mu = u''_\mu$ for $\mu \neq \nu$, $|u'_\nu - u''_\nu| \leqq \delta \pmod{2 N \pi}$ and v in R. Let $H(\underline{u};v) = T_{\nu;N,\delta}\, G(\underline{u};v)$.

If $\underline{u}$ is an arbitrary vector, then there exists a vector $\underline{u}^*$ and an integer q satisfying

$u^*_\mu = u_\mu$ for $\mu \neq \nu$, and
$u_\nu - u^*_\nu = 2\, q\, N\, \pi$ where $0 \leqq u^*_\nu \leqq 2\, N\, \pi$.

We then have

$$H(\underline{u};v) = H(\underline{u}^*;v), \text{ and}$$
$$(G(\underline{u};v), G(\underline{u}^*;v)) \ \varepsilon\ F_{d,\epsilon/2} \cap F_{p,\eta}$$

for all v in R.

Now to establish (7) there are two cases:

(a) $0 \leqq u^*_\nu \leqq 2\, N\, \pi - \delta$. In this case we have $H(\underline{u}^*;v) = G(\underline{u}^*;v)$ and hence

$$d(H(\underline{u};v),\ G(\underline{u};v)) \leqq d(H(\underline{u};v),\ H(\underline{u}^*;v)) \tag{10}$$
$$+\ d(G(\underline{u}^*;v),\ G(\underline{u};v)) \leqq 0 + \epsilon/2 \leqq \epsilon$$

since $|u_\nu^* - u_\nu| = 0 \pmod{2N\pi}$.

(b) $2N\pi - \delta < u_\nu^* \leqq 2N\pi$. In this case we have by (9) that

$$d(G_{\nu;2N\pi-\delta}(\underline{u};v),\ G(\underline{u};v)) \leqq \epsilon/2, \tag{11}$$

and by (8) that

$$d(G_{\nu;2N\pi-\delta}(\underline{u};v),\ H(\underline{u};v)) \leqq \epsilon/2 \tag{12}$$

since $p(G_{\nu;2N\pi-\delta}(\underline{u};v),\ G_{\nu;0}(\underline{u};v)) \leqq \eta$ by (9). Combining (11) and (12) we have

$$d(H(\underline{u};v),\ G(\underline{u};v)) \leqq \epsilon \text{ for all } v \text{ in } R. \tag{13}$$

Thus taking (10) and (13) together it follows that

$$(H(\underline{u};v),\ G(\underline{u};v)) \in F_{d,\epsilon} \subset U$$

for every vector $\underline{u}$ and every v in R.

Theorem 5. Let $G(\underline{u};v)$ be a uniformly continuous family of limit periodic functions and let U be an arbitrary member of the uniformity $\underline{U}$. Then there exist two positive integers m and N and a continuous function $g(\underline{u};v) = g(u_1,\ldots,u_m;v)$ with the period $2N\pi$ in each variable such that

$$(g(\underline{u};v),\ G(\underline{u};v)) \in U$$

for every vector $\underline{u}$ and every v in R.

Proof. There exists a $F_{d,\epsilon} \subset U$. According to Theorem 14 of Section IIIB there exists an integer m and a uniformly continuous family of limit periodic functions $g_o(\underline{u};v) = g_o(u_1,\ldots,u_m;v)$ such that

$$d(g_o(\underline{u};v),\ G(\underline{u};v)) \leqq \epsilon/2$$

for all vectors $\underline{u}$ and all v in R. To prove the theorem we need only construct a finite sequence $g_o(\underline{u};v),\ldots,g_m(\underline{u};v)$ of uniformly continuous families of limit periodic functions satisfying the conditions:

(a) $d(g_\nu(\underline{u};v),\ g_{\nu-1}(\underline{u};v)) \leqq \epsilon/2m,\quad \nu=1,\ldots,m,$ for every vector $\underline{u}$ and all v in R.

(b) $g_\nu(\underline{u};v)$ is independent of the variables $u_{m+1}, u_{m+2},\ldots$.

(c) There exists a sequence $N_1,\ldots,N_m$ of positive integers such that $g_\nu(\underline{u};v)$ $\nu=1,\ldots,m$ has the periods $2N_1\pi,\ldots,2N_\nu\pi$ in the variables $u_1,\ldots,u_\nu$.

In fact, $g_m(\underline{u};v)$ would then depend only on $u_1,\ldots,u_m$, would have the period $2N_1\ldots N_m\pi$ in each of these variables, and would satisfy the condition

$$d(g_m(\underline{u};v),\ G(\underline{u};v)) \leqq \sum_{\nu=1}^{m} d(g_\nu(\underline{u};v),\ g_{\nu-1}(\underline{u};v))$$

$$+\ d(g_o(\underline{u};v),\ G(\underline{u};v)) \leqq m\cdot\epsilon/2m + \epsilon/2 = \epsilon,$$

for all vectors $\underline{u}$ and all v in R. Thus we have only to construct the finite sequence of functions $g_o(\underline{u};v),\ldots,g_m(\underline{u};v)$

satisfying the conditions (a), (b), and (c). For each $\nu=1,\dots,m$ we choose

$$g_{\nu}(\underline{u};v) = T_{\nu;N_{\nu},\delta}\ g_{\nu-1}(\underline{u};v)$$

such that condition (a) is satisfied. According to Theorem 4 this is always possible. From Theorem 2 it follows that $g_{\nu}(\underline{u};v)$ is a uniformly continuous family of limit periodic functions, and from Theorem 3 it follows that the conditions (b) and (c) are satisfied. This completes the proof of Theorem 5.

Theorem 6. (<u>The Approximation Theorem</u>). Let $(X,\underline{U})$ be a complete, continuously locally arcwise connected uniform space. To a uniformly continuous family of almost periodic movements $x = f(t;v)$, $-\infty < t < \infty$, $v \in R$, $x \in X$, and to each member U of the uniformity $\underline{U}$ correspond positive integers m and N, a continuous function $g(u_1,\dots,u_m;v)$, $-\infty < u_{\nu} < \infty$, $\nu = 1,\dots,m$, $v \in R$ with the period $2N\pi$ in each of the variables u_{ν}, $\nu = 1,\dots,m$, and linearly independent numbers $\beta_1,\dots,\beta_m$ such that

$$(g(\beta_1 t,\dots,\beta_m t;v),\ f(t;v)) \in U$$

for all real t and all v in R.

Proof. By Theorem 12 of Section IIIB there corresponds to $f(t;v)$ a set of linearly independent numbers $\beta_1, \beta_2,\dots$ (a basis of $f(t;v)$), and by Theorem 13 of Section IIIB there corresponds to $f(t;v)$ and its basis $\underline{\beta}$ a uniformly continuous family of limit periodic functions $G(\underline{u};v)$ (a spatial extension of $f(t;v)$) such that

$$f(t;v) = G(\underline{\beta}t;v) \ . \tag{14}$$

Furthermore by Theorem 5 of this section there correspond to $G(\underline{u};v)$ and U two positive integers m and N and a continuous function $g(\underline{u};v) = g(u_1,\ldots,u_m;v)$ with the period $2\,N\,\pi$ in each variable such that

$$(g(\underline{u};v),\ G(\underline{u};v)) \ \varepsilon \ U \tag{15}$$

for every vector $\underline{u}$ and every v in R. Combining (14) and (15) we have

$$(g(\underline{\beta}t;v),\ f(t;v)) \ \varepsilon \ U$$

for all real t and all v in R. This completes the proof of Theorem 6, and concludes Chapter III of the thesis.

IV. ON ALMOST PERIODIC SOLUTIONS OF A CLASS OF DIFFERENTIAL EQUATIONS

Let E_N denote N-dimensional complex vector space with vectors x whose components with respect to a given basis are denoted by

$$x = (x^i) = \begin{bmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^N \end{bmatrix}.$$

Let $\| x \|$ denote the norm of x defined by the equation

$$\| x \| = \sum_{i=1}^{N} | x^i | .$$

This norm satisfies the usual rules

$$\begin{aligned} &(a)\ \| x \| \geqq 0,\ \| x \| = 0 \text{ only if } x = 0 \text{ (zero vector)}, \\ &(b)\ \| \alpha x \| = | \alpha | \cdot \| x \| ,\ \alpha \text{ complex}, \\ &(c)\ \| x + y \| \leqq \| x \| + \| y \| . \end{aligned} \tag{1}$$

Furthermore, if $x(t)$ is a vector-valued function of a real variable t, we define as is usual

$$dx/dt = (dx^i/dt) \text{ and } \int x(t)dt = (\int x^i(t)dt) .$$

The following well-known inequality will be useful:

$$\left\| \int_a^b x(t)\, dt \right\| \leqq \int_a^b \| x(t) \|\, dt, \qquad b \geqq a.$$

We note that E_N with the norm $\|x\|$ is a separable metric space with metric

$$d(x,y) = \| x - y \| ,$$

and that convergence of a sequence of vectors with respect to this metric is equivalent to the convergence of the individual components of the vectors.

A continuous vector-valued function $x(t)$ of a real variable t is said to be an <u>almost periodic vector function</u> if to each positive ϵ there corresponds a relatively dense set of real numbers τ such that

$$\| x(t + \tau) - x(t) \| \leqq \epsilon \quad \text{for all real } t.$$

Since a finite number of complex-valued almost periodic functions have, for each positive ϵ, a relatively dense set of ϵ-translation numbers in common, it follows from the equation

$$\| x(t + \tau) - x(t) \| = \sum_{i=1}^{N} | x^i(t + \tau) - x^i(t) |$$

that $x(t)$ is an almost periodic vector function if and only if each of its components $x^i(t)$ is an almost periodic function.

Similarly, a vector-valued function $x(n)$ defined on the integers, $n = 0, \pm 1, \pm 2, \ldots$, is called an <u>almost periodic vector sequence</u> if to each positive ϵ there corresponds a relatively dense set of integers τ such that

$$\| x(n+\tau) - x(n) \| \leqq \epsilon \quad \text{for } n = 0, \pm 1, \pm 2, \ldots .$$

Definition 1. Let $F(x,t)$ denote a vector-valued function defined for all real t and all vectors x in some connected open subset D of E_N such that

(a) $F(x,t)$ is an almost periodic vector function of t for each x in D, and

(b) $F(x,t)$ satisfies the following Lipschitz condition for all x and y in D.

$$\| f(x,t) - F(y,t) \| \leqq K \| x - y \| \quad \text{for all real } t.$$

$F(x,t)$ can be thought of as a family of almost periodic movements in the metric space E_N. For this purpose we shall adopt H. Tornehave's definition of a "uniformly continuous family of almost periodic movements". (Cf. [11], Definition 2.)

Definition 2. Let S denote a compact separable metric space. Let M denote an arbitrary metric space with metric d. A function $f(t;v)$ defined for all real t and all v in S, and with values in M, is called a _uniformly continuous family of almost periodic movements_ when

(a) the function $f(t;v)$ is almost periodic for every fixed v in S, and

(b) to each positive ϵ and v_o in S there corresponds a neighborhood $U_\epsilon(v_o)$ of v_o such that $d(f(t;v), f(t;v_o)) \leqq \epsilon$ for all real t and all v in $U_\epsilon(v_o)$.

Lemma 1. Let C denote a compact subset of D, where D is the connected open subset of E_N on which $F(x,t)$ is defined. Then $F(x,t)$, with x restricted to C, is a uniformly continuous family of almost periodic movements in E_N.

Proof. Condition (a) of Definition 2 is the same as condition (a) of Definition 1.

To each positive ϵ and each x_o in C choose

$$U_\epsilon(x_o) = \{ x : \| x - x_o \| \leqq \epsilon/K \} .$$

Then by condition (b) of Definition 1 we have

$$\| F(x,t) - F(x_o,t) \| \leqq K \| x - x_o \| \leqq K \epsilon/K = \epsilon$$

for all x in $U_\epsilon(x_o)$ and all real t. Hence condition (b) of Definition 2 is satisfied. This completes the proof.

Hans Tornehave has proved the following result concerning a uniformly continuous family of almost periodic movements. (Cf. [11], Theorem 1.)

Lemma 2. To each uniformly continuous family $f(t;v)$ of almost periodic movements in a metric space M there corresponds a real-valued almost periodic function $g(t)$ such that, for each positive ϵ, the set of ϵ-translation numbers common to all the functions of the family $f(t;v)$ contains the set of ϵ-translation numbers of $g(t)$.

This result of Hans Tornehave will be used here to obtain a necessary and sufficient condition that a given vector solution $\phi(t)$ of the differential equation

$$dx/dt = F(x,t) \qquad -\infty < t < \infty \tag{2}$$

be an almost periodic vector function. The following fact will also be needed.

Lemma 3. If $x(n)$, $n = 0, \pm 1, \pm 2, \ldots$, is an almost periodic vector sequence and $g(t)$ is a real-valued almost periodic function, then for each positive ϵ the set of ϵ-translation integers common to $x(n)$ and $g(t)$ is a relatively dense set.

Proof. The proof of this lemma is analogous to the proof that the sum of two almost periodic sequences is again an almost periodic sequence. However, as the necessary changes in the details are not completely obvious, a detailed proof of Lemma 3 is given here.

Consider S. Bochner's <u>translation function</u> $e(n)$, $n = 0, \pm 1, \pm 2, \ldots$, (cf. Besicovitch [6], pp. 8-9) defined by the equation

$$e(n) = \sup_{m=0,\pm1,\pm2,\ldots} \| x(m + n) - x(m) \| .$$

The following three properties of this function are easy consequences of its definition and the properties (1) of the norm:

(a) $e(n) \geqq 0$, $e(0) = 0$,

(b) $e(-n) = e(n)$,

(c) $e(n + m) \leqq e(n) + e(m)$.

Furthermore e(n) is an almost periodic sequence. To prove this we write

$$e(n+\tau) \leqq e(n) + e(\tau), \text{ and}$$
$$e(n) \leqq e(n+\tau) + e(-\tau) = e(n+\tau) + e(\tau),$$

and hence

$$\left| e(n+\tau) - e(n) \right| \leqq e(\tau) \quad \text{for all integers } n \text{ and } \tau.$$

For n = 0 we have

$$\left| e(\tau) - e(o) \right| = e(\tau),$$

and so

$$\sup_{n=0,\pm1,\pm2,\ldots} \left| e(n+\tau) - e(n) \right| = e(\tau).$$

That is,

$$\sup_{n=0,\pm1,\pm2,\ldots} \left| e(n+\tau) - e(n) \right| = \sup_{m=0,\pm1,\pm2,\ldots} \left\| x(m+\tau) - x(m) \right\| .$$

From this last equation we can conclude that the set of ϵ-translation integers τ of e(n) is identical to the set of ϵ-translation integers of x(n). Since the last mentioned set is relatively dense for each positive ϵ, e(n) is an almost periodic sequence.

Thus to prove the lemma, we need only to show that the set of ϵ-translation integers common to g(t) and e(n) is a relatively dense set. To this end let τ_g and τ_e denote translation integers of $g(t)$ and $e(n)$, respectively. Let l_o be a positive integer such that every interval $[s, s + l_o]$, $s = 0, \pm 1, \pm 2, \ldots$, contains an $\epsilon/2$-translation integer τ_g and an $\epsilon/2$-translation integer τ_e. Let F_n denote the interval $[nl_o, nl_o + l_o]$ for $n = 0, \pm 1, \pm 2, \ldots$. In each F_n select two $\epsilon/2$-translation integers, $\tau_e^{(n)}$ and $\tau_g^{(n)}$. The differences $d^{(n)} = \tau_e^{(n)} - \tau_g^{(n)}$ satisfy

$$- l_o \leqq d^{(n)} \leqq l_o \quad \text{for every } n.$$

Those of the $2\,l_o + 1$ integers $- l_o, \ldots, l_o$ which occur among the differences $d^{(n)}$ for some n we shall denote by $\{ i_1, i_2, \ldots, i_p \}$, $1 \leqq p \leqq 2\,l_o + 1$. For each i_ν, $\nu = 1, 2, \ldots, p$, there exists an integer r_ν with smallest absolute value $|r_\nu|$ such that $d^{(r_\nu)} = i_\nu$. Let

$$R = \max \{ |r_\nu| \; : \; \nu = 1, 2, \ldots, p \}.$$

Choose an integer $L > l_o$ so large that the interval $[-L, L]$ contains the interval F_R. Since $L > l_o$, the interval $[S - L, S + L]$ contains at least one of the intervals F_n (of length l_o), say F_r. Select from the intervals F_n contained in the interval $[-L, L]$ the one interval F_q for which $d^{(q)} = d^{(r)}$. Then

$$(\tau_e^{(r)} - \tau_e^{(q)}) - (\tau_g^{(r)} - \tau_g^{(q)}) = d^{(r)} - d^{(q)} = 0.$$

Hence, since the sum (or difference) of $\epsilon/2$-translation numbers is an ϵ-translation number, if we set $\tau^* = \tau_e^{(r)} - \tau_e^{(q)} = \tau_g^{(r)} - \tau_g^{(q)}$ we have

$$\tau^* \in \{\tau_{e(n)}(\epsilon)\} \cap \{\tau_{g(t)}(\epsilon)\} , \qquad (3)$$

and since $-L \leqq \tau_e^{(q)} \leqq L$ and $s - L \leqq \tau_e^{(r)} \leqq s + L$ we have

$$s - 2L \leqq \tau^* \leqq s + 2L. \qquad (4)$$

Since s is an arbitrary integer, (3) and (4) prove the lemma.

Theorem 1. If $f(t;v)$ is a uniformly continuous family of almost periodic movements in a metric space M, and $x(n)$ is an almost periodic vector sequence, then for each positive ϵ the set of ϵ-translation integers common to $x(n)$ and all $f(t;v)$ is a relatively dense set.

Proof. By Lemma 3

$$\{\tau_{x(n)}(\epsilon)\} \cap \{\tau_{g(t)}(\epsilon)\} \text{ is r. d. },$$

and by Lemma 2

$$\{\tau_{g(t)}(\epsilon)\} \subset \bigcap_{v \in S} \{\tau_{f(t;v)}(\epsilon)\} .$$

Consequently we have

$$\{\tau_{x(n)}(\epsilon)\} \cap \{\tau_{g(t)}(\epsilon)\} \subset \{\tau_{x(n)}(\epsilon)\} \cap \left(\bigcap_{v \in S} \{\tau_{f(t;v)}(\epsilon)\}\right) .$$

But a set containing a relatively dense subset is relatively dense. This completes the proof.

The proof of the main theorem makes use of the following inequality implication.

Lemma 4. Let $x(t)$ be a vector-valued function defined and continuous for all real t in an interval $a \leqq t \leqq b$. If

$$\| x(t) \| \leqq A + B \int_a^t \| x(s) \| \, ds \qquad a \leqq t \leqq b \tag{5}$$

where A and B are positive constants, then

$$\| x(t) \| \leqq Ae^{B(b - a)} \qquad a \leqq t \leqq b .$$

Proof. Let $R(t)$ denote $\int_a^t \| x(s) \| ds$. Then $R(a) = 0$ and $R'(t) = \| x(t) \|$, and hence (5) becomes

$$R'(t) - B \cdot R(t) \leqq A, \qquad a \leqq t \leqq b. \tag{6}$$

Multiplying (6) by e^{-Bt} and integrating from a to t we have

$$R(t)e^{-Bt} \leqq A/B(e^{-aB} - e^{-Bt}), \text{ or}$$

$$R(t) \leqq A/B \, (e^{B(t - a)} - 1) . \tag{7}$$

Finally, combining (5) and (7), we have

$$\| x(t) \| \leqq A + B \cdot R(t) \leqq e^{B(t - a)} \leqq e^{B(b - a)} ,$$
$$a \leqq t \leqq b .$$

Theorem 2. (Main Theorem). Let $\phi(t)$ be a vector solution of Differential Equation (2) for all real t and let D (cf. Definition 1) contain the closure of the range of $\phi(t)$. Then a necessary and sufficient condition that $\phi(t)$ be an almost periodic vector function is that $\phi(n)$, $n = 0, \pm 1, \pm 2, \ldots$, be an almost periodic vector sequence.

Proof. The necessity is trivial since every almost periodic vector function is an almost periodic vector sequence when restricted to the integers. This fact is an immediate consequence of Theorem 1 of Section IIA and the fact that the set of ϵ-translation numbers of $\phi(t)$ is identical to the set of ϵ-translation numbers of Bochner's translation function (cf. Besicovitch [6], pp. 8-9) $e(t) = \sup_{-\infty<s<\infty} \|\phi(s + t) - \phi(s)\|$.

Thus it is the sufficiency of the condition with which we are primarily concerned here. Consequently for the remainder of the proof we assume that $\phi(n)$ is an almost periodic vector sequence and that

$$d/dt\ \phi(t) = F(\phi(t), t) \qquad -\infty<t<\infty\ , \tag{8}$$

where $F(x,t)$ is the family of almost periodic vector functions defined in Definition 1. With these hypotheses we intend to show that $\phi(t)$ is an almost periodic vector function.

First of all we shall prove that the range of $\phi(t)$ is a bounded set in E_N.

$$\begin{aligned} \|\phi(t)\| &\leq \|\phi(t) - \phi(n)\| + \|\phi(n)\| \\ &\leq \|\phi(t) - \phi(n)\| + \sup_m \|\phi(m)\|\ . \end{aligned} \tag{9}$$

$\sup_m \| \phi(m) \|$ is finite since every almost periodic vector sequence is bounded.

$$\begin{aligned} \| F(\phi(n),t) \| &\leqq \| F(\phi(n),t) - F(\phi(o),t) \| + \| F(\phi(o),t) \| \\ &\leqq K \| \phi(n) - \phi(o) \| + \sup_t \| F(\phi(o),t) \| \qquad (10) \\ &\leqq K(\sup_n \| \phi(n) \| + \| \phi(o) \|) + \sup_t \| F(\phi(o),t) \| . \end{aligned}$$

Let M denote $K(\sup_n \| \phi(n) \| + \| \phi(o) \|) + \sup_t \| F(\phi(o),t) \|$. Then (10) becomes

$$\| F(\phi(n),t) \| \leqq M \quad \text{for all integers } n \text{ and real } t. \qquad (11)$$

Since the Differential Equation (8) can be written as the integral equation

$$\phi(t) = \phi(o) + \int_o^t F(\phi(s),s)\, ds ,$$

we can use (11) and the Lipschitz condition on $F(x,t)$ to obtain the following inequality which holds for all t in the interval $n \leqq t \leqq n + 1$.

$$\begin{aligned} \| \phi(t) - \phi(n) \| &\leqq \left\| \int_n^t F(\phi(s),s)ds - \int_n^t F(\phi(n),s)ds \right\| \\ &\quad + \left\| \int_n^t F(\phi(n),s)ds \right\| \\ &\leqq \int_n^t \| F(\phi(s),s) - F(\phi(n),s) \| ds + M \qquad (12) \\ &\leqq K \int_n^t \| \phi(s) - \phi(n) \| ds + M. \end{aligned}$$

From (12) it follows by Lemma 4 that

$$\| \phi(t) - \phi(n) \| \leqq Me^{K} \quad \text{for } n \leqq t \leqq n + 1 \, . \tag{13}$$

Combining (9) and (13) we now have

$$\| \phi(t) \| \leqq Me^{K} + \sup_{m} \| \phi(m) \| \; , \quad n \leqq t \leqq n + 1$$

for each integer n. Thus $\phi(t)$ is a bounded vector function for all real t.

But in E_N the closure of a bounded set is bounded and since compact sets in E_N are precisely those sets which are both closed and bounded it follows from Lemma 1 that $F(x,t)$, with x restricted to the closure of the range of $\phi(t)$, is a uniformly continuous family of almost periodic movements in E_N. The range of $\phi(t)$ will be denoted by R, and its compact closure by $\overline{R}$. Now by Theorem 1 there exists, for each positive ϵ, a relatively dense set E of ϵ-translation integers τ common to $\phi(n)$ and all the functions of the family $\{F(x,t) : x \in \overline{R}\}$. We are now in a position to show that $\phi(t)$ is an almost periodic vector function.

$\phi(t)$ is certainly continuous for all t since we have assumed it to be differentiable for all t. Indeed, we have already used this fact in applying Lemma 4 to the inequality (12). Thus it only remains to be shown that $\phi(t)$ possesses a relatively dense set of ϵ-translation numbers corresponding to each positive ϵ. Consider the following inequality:

$$\| \phi(t+\tau) - \phi(t) \| = \| \left(\int_0^{t+\tau} - \int_0^t \right) F(\phi(s),s)ds \|$$

$$= \| \left(\int_0^{n+\tau} - \int_0^n + \int_{n+\tau}^{t+\tau} - \int_n^t \right) F(\phi(s),s)ds \|$$

$$\leqq \| \phi(n+\tau) - \phi(n) \| \tag{14}$$

$$+ \int_n^t \| F(\phi(s+\tau), s+\tau) - F(\phi(s+\tau),s) \| \, ds$$

$$+ \int_n^t \| F(\phi(s+\tau),s) - F(\phi(s),s) \| \, ds \, .$$

If we now let τ be an ϵ-translation integer of the relatively dense set E of ϵ-translation integers common to $\phi(n)$ and all $F(x,t)$ for x in $\overline{R}$, the inequality (14) becomes

$$\| \phi(t+\tau) - \phi(t) \| \leqq \epsilon + \epsilon(t-n) + K \int_n^t \| \phi(s+\tau) - \phi(s) \| ds$$

$$\leqq 2\epsilon + K \int_n^t \| \phi(s+\tau) - \phi(s) \| \, ds,$$

for all t in the interval $n \leqq t \leqq n+1$. But then by Lemma 4 we have

$$\| \phi(t+\tau) - \phi(t) \| \leqq 2\epsilon e^K \qquad n \leqq t \leqq n+1,$$

for every integer n. Thus we see that an ϵ-translation integer common to $\phi(n)$ and all $F(x,t)$ for x in $\overline{R}$ is a $2\epsilon e^K$-translation integer for $\phi(t)$. Therefore $\phi(t)$ is an almost periodic vector function. This completes the proof of Theorem 2.

V. SUMMARY

In this thesis a generalization of Hans Tornehave's theory [11] of almost periodic movements in metric spaces is presented. Tornehave considers functions f(t) which are defined for all real t, and have their values in a complete metric space (X,d). The generalization presented in this thesis consists in replacing the metric space (X,d) by the more general concept of a uniform space ($X,\underline{U}$). A weaker definition of completeness than is usually adopted for uniform spaces proved to be sufficient for this generalization. The definition adopted here is as follows: a uniform space is called complete if and only if every closed and totally bounded subset is compact. Only those parts of the generalization are given which are necessary to establish the analogue of Tornehave's approximation theorem.

Finally, one of Tornehave's theorems, concerning uniformly continuous families of almost periodic movements in metric spaces, is used to obtain a theorem concerning almost periodic solutions of a vector differential equation of the form

$$dx/dt = F(x,t) \ . \qquad (1)$$

Here F(x,t) is assumed to be almost periodic in t for each x, and Lipschitz in x uniformly in t. The theorem proved can be stated as follows: a necessary and sufficient condition that a vector solution $\phi(t)$ of (1) be almost periodic is that $\phi(n)$, $n = 0, \pm 1, \pm 2, \ldots$, be an almost periodic sequence.

VI. REFERENCES

1. Bohr, Harald. 1924, 1925, 1926. Zur Theorie der fastperiodischen Funktionen. I, II, III. Acta Math. 45, 29-127; 46, 101-214; 47, 237-281.

2. Besicovitch, A. S. 1926. On generalized almost periodic functions. Proc. Land. Math. Soc. series 2, 25, 495-512.

3. Stepanoff, W. 1926. Uber einigen Verallgemeinerungen der fastperiodischen Funktionen. Math. Ann. 95, 473-498.

4. Weyl, H. 1926. Integralgleichungen und fastperiodischen Funktionen. Math. Ann. 97, 338-356.

5. Seynsche, Ingeborg. 1931. Zur Theorie der fastperiodischen Zahlfolgen. Rendiconti Circolo Matematico di Palermo 55, 395-421.

6. Besicovitch, A. S. 1932. Almost periodic functions. Dover Publications, Inc., New York (1954). (This is a reissue of the first edition published by Cambridge University Press in 1932.)

7. Bochner, S. and von Neumann, J. 1935. Almost periodic functions in groups, II. Trans. Amer. Math. Soc. 37, 21-50.

8. Weil, A. 1937. Sur les espaces a structure uniforme et sur la topologie generale. Actualities Sci. Ind. no. 551.

9. Mackey, George W. 1946. On convex topological linear spaces. Trans. Amer. Math. Soc. 60, 519-537.

10. Bohr, Harald. 1952. Collected Mathematical Works, volumes 2 and 3. Danish Math. Soc., Copenhagen.

11. Tornehave, Hans. 1954. On almost periodic movements. Dan. Mat. Fys. Medd. 28 no. 13.

12. Kelley, John L. 1955. General Topology. D. Van Nostrand Company, Inc., New York.

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